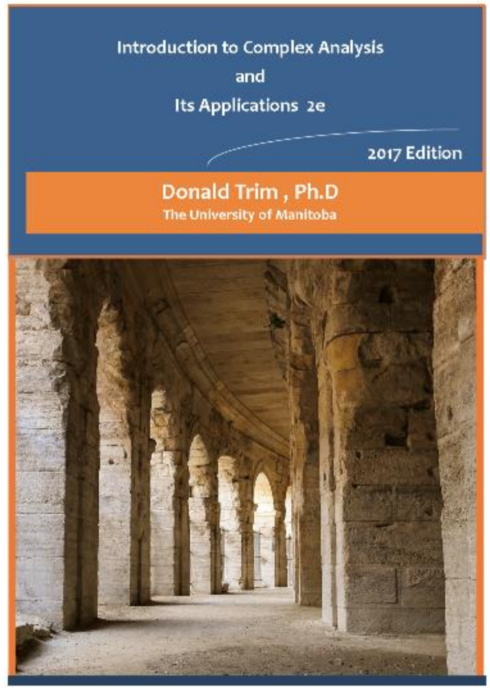
Book Information

Just Published! 2017 copyright!



2nd edition just published: 2017 copyright.

Introduction to Complex Analysis and Its Applications 2e

Donald Trim, Ph.D.
The University of Manitoba

Paperback: ***$44.95***

Student Solutions Manual ***$11.95***

Instructor Supplements: **Solutions Manual**
Student Supplements: **Student Solutions Manual**

3 about Trim 2e:

1. Previously published by PWS Publishing, 2e is for undergraduate students in science and engineering.
2. Can be used in one-semester or two-semester courses
3. The presentation assumes a working knowledge of single-variable real calculus, including infinite series, and familiarity with partial differentiation and line integrals. Discussions are rigorous, but the approach is intuitive.

- This text explores the intimate relationships among the three major topics of complex calculus — differentiation, integration, and infinite series. Properties in any one of these topics are reflected in properties in the other two, so much so, that one could commence complex calculus with any one of the three topics. The preference in this text is to begin with differentiation, it being, perhaps, the easiest of the three topics for most students in real calculus. This is then followed with integration and infinite series.
- Geometric visualizations are used to introduce and clarify ideas whenever possible.
- The study of calculus of complex functions can be a very rewarding experience. Topics unfold naturally, and each new topic intimately relates to everything that has gone before. Proofs of most results, even the very profound, are usually quite straightforward, and the material is rich in applications.
- A Student Solution Manual is also available for the text. This manual contains solutions to even-numbered exercises in the text. Solutions are sufficiently detailed that the reader should have no difficulty following the logic.

Table of Contents and Sample Chapter

About this Publication:

This text... is written for undergraduate students in science and engineering. At a brisk pace, theory and a selection of applications can be covered in one semester; for complete coverage at a more leisurely pace allow two semesters. The presentation assumes a working knowledge of single-variable real calculus, including infinite series, and familiarity with partial differentiation and line integrals. Discussions are rigorous, but the approach is intuitive.

We take every opportunity to compare properties of complex functions to those of real functions. Only then can the power and elegance of complex variable theory truly be appreciated. Plenty of exercises, with answers, are provided. Exercises are graded from routine, for reinforcement of fundamentals, to challenging, for the more enterprising reader.

True understanding cannot be achieved simply by reading the text. You must engross yourself in the subject and we have included numerous exercises for this purpose. Try as many as you can; you will not regret the effort.

Finally, we wish you every success in your studies, and we hope that you will share in our fascination with the subject.

For Instructors: A complete Solutions Manual containing detailed solutions to chapter problems is available.

For Students: A Student Solution Manual containing solutions to half of the text's problems is available for purchase by students.

Introduction to Complex Analysis

and

Its Applications

Donald Trim

The University of Manitoba

Winnipeg, Manitoba

Canada, R3T 2N2

For more information, contact

Textbook Media Press
1808 Dayton Avenue
Saint Paul, MN 55104

Or you can visit our Internet site at

http://www.textbookmedia.com

or write

info@textbookmedia.com

For permission to use material from this text
or product, submit a request online at
info@textbookmedia.com

Introduction to Complex Analysis and Its Applications 2e

2017 Edition

Donald Trim

13 Digit ISBN: 978-0-9969962-8-0

Textbook Media Press is a Minnesota-based educational publisher.

We deliver textbooks and supplements with the quality instructors expect,
while providing students with media options at uniquely affordable prices.

TABLE OF CONTENTS

Chapter 10 The z-Transform

Appendix A Answers to Exercises

PREFACE

To the Instructor

This text is written for undergraduate students in science and engineering. At a brisk pace, theory and a selection of applications can be covered in one semester; for complete coverage at a more leisurely pace allow two semesters. The presentation assumes a working knowledge of single-variable real calculus, including infinite series, and familiarity with partial differentiation and line integrals. Discussions are rigorous, but the approach is intuitive.

Chapter 1 introduces complex numbers. We add, subtract, multiply, divide, and find roots of complex numbers. We introduce the Cartesian, polar, and exponential forms for complex numbers, illustrate complex numbers geometrically, and solve polynomial equations. Chapter 2 begins with the algebra of complex functions, discusses limits and continuity, and ends with complex differentiation and analyticity. The complex exponential, trigonometric, hyperbolic, logarithmic, and inverse trigonometric and hyperbolic functions are introduced in Chapter 3. We concentrate on algebraic properties of these functions, and their zeros and singularities, but, in preparation for conformal mapping in Chapter 7, we also stress the importance of visualizing a function $w = f(z)$ as a mapping from the z-plane to the w-plane. The fact that real and imaginary parts of analytic functions are harmonic suggests the use of complex functions to problems in electrostatics, steady-state temperature distributions, and fluid flows. These applications are introduced in Chapter 3.

In Chapter 4 we focus on contour integration. We verify the Cauchy-Goursat theorem, introduce Cauchy's integral formulas and Poisson's integral formulas, and develop modulus principles for complex and harmonic functions. Power series and Laurent series with the subsequent classification of singularities of complex functions are discussed in Chapter 5. Chapters 4 and 5 culminate in the theory of residues in Chapter 6. The power of residues is brought out in the evaluation of real trigonometric and improper integrals, the principle of the argument, Rouche's theorem, and summations of certain types of infinite series.

Although the first six chapters emphasize algebraic consequences of analyticity, and geometric results in the context of conformal mapping are discussed in Chapter 7, we constantly stress the geometric interpretation of a function as a mapping. Having students repeatedly visualize regional mappings by exponential, trigonometric, and hyperbolic functions, and their inverses, we pave the way for the application of conformal mappings to problems in heat flow, fluid flow, and electrostatics. These applications are dealt with in Chapter 7, perhaps with more detail than most texts at this level, but they are introduced in Chapter 3 as illustrations of the fact that families of curves defined by harmonic conjugates are orthogonal trajectories. In this way we give students an early indication that complex functions are indeed useful in physical applications.

In Chapter 8, we discuss Laplace transforms. We develop their elementary properties and show how they can be used to replace ordinary differential equations with algebraic equations. Cauchy's integral formula and residues then lead to a direct formula for inverting Laplace transforms. The transform is used to solve partial differential equations on bounded and semi-infinite domains. Finally, we introduce transfer functions and the notion of stability for systems governed by differential equations.

In Chapter 9, we use the Fourier transform, the Fourier sine transform, and the Fourier cosine transform to reduce partial differential equations on infinite and semi-infinite intervals to ordinary differential equations. We apply the technique to heat conduction, vibration, and potential problems.

In Chapter 10, we discuss the z-transform. The discussion parallels that for the Laplace transform in that we discuss properties of the transform, followed by its use in solving first- and second-order difference equations. Applications are discussed for difference equations as are transfer functions and stability for discrete systems.

We take every opportunity to compare properties of complex functions to those of real functions. Only then can the power and elegance of complex variable theory truly be appreciated. Plenty of exercises, with answers, are provided. Exercises are graded from routine, for reinforcement of fundamentals, to challenging, for the more enterprising reader. A complete solutions manual containing detailed solutions to all problems is available.

To the Student

When the domain and range of a function are sets of complex numbers, we speak of a complex function of a complex variable. This text is an introduction to the properties and applications of such functions; in particular, we develop the *calculus* of complex functions. Your earliest calculus studies involved real-valued functions $f(x)$ of a real variable x. You found their derivatives and integrals, and used these in many geometric and physical applications. Your second exposure to calculus expanded single-variable concepts to functions of many real variables $f(x, y, \ldots)$. Once again you differentiated and integrated these functions, and applied the derivatives and integrals in more complicated, but more realistic applications. It is now time to take the third, and for most, the final step; calculus of functions $f(z)$ of a complex variable z.

You are well aware of the connection between integration and differentiation in real calculus; you will be amazed at the intimate relationships among the three major topics of complex calculus — differentiation, integration, and infinite series. Properties in any one of these topics are reflected in properties in the other two, so much so, that we could commence complex calculus with any one of the three topics. Our preference is to begin with differentiation, it being, perhaps, the easiest of the three topics for most students in real calculus. We follow this with integration and infinite series.

Calculus of complex functions has many similarities to both single-variable (real) calculus and multivariable (real) calculus, and we shall point these out as discussions unfold. But there are also striking differences, and we shall be even more careful to draw these to your attention. For instance, in most applications of real calculus, we are concerned with points where functions are well-behaved. Contrarily, points where complex functions misbehave are often the most valuable in applications. Existence of the first derivative of a real function implies nothing about existence of higher order derivatives, whereas existence of the first derivative of a complex function implies existence of derivatives of all orders.

Complex calculus provides proofs to many results that seem otherwise intractable in real analysis; it also provides a clearer understanding to some topics in real analysis. For example, the often assumed result that every real polynomial has a zero is verified in Chapter 4; why there is a number called the "radius" of convergence of a real power series becomes clear when we study complex series in Chapter 5; and why solutions of Laplace's equation do not have relative extrema is verified in Chapter 4.

We use geometric visualizations to introduce and clarify ideas whenever possible. Visualizations of real functions $f(x)$ as curves and $f(x, y)$ as surfaces are unavailable for complex functions $f(z)$. Instead, we interpret $f(z)$ as a mapping from one complex plane to another. The mapping approach helps us appreciate many of the properties of functions, and in addition, paves the way for the important topic of conformal mapping in Chapter 7 with its applications to problems in electrostatics, heat flow, and fluid flow.

The study of calculus of complex functions can be a very rewarding experience. Topics unfold naturally, and each new topic intimately relates to everything that has gone before. Proofs of most results, even the very profound, are usually quite straightforward, and the material is rich in applications. To aid in recognizing when discussions are complete, we have designated the end of the proof of a theorem by ▮, and the end of an example by a •.

True understanding cannot be achieved simply by reading the text. You must engross yourself in the subject and we have included numerous exercises for this purpose. Try as many as you can; you will not regret the effort. Answers to exercises can be found in Appendix A. A solutions manual with detailed solutions to all problems is available.

Finally, we wish you every success in your studies, and we hope that you will share in our fascination with the subject.

CHAPTER 1 COMPLEX NUMBERS

When the domain and range of a function are sets of complex numbers, we speak of a complex function of a complex variable. In this chapter we study complex numbers in preparation for complex functions. We add, subtract, multiply, and divide complex numbers, and find their roots. We introduce the Cartesian, polar, and exponential forms of complex numbers, illustrate complex numbers in the complex plane, and solve polynomial equations.

§1.1 Introduction

Perhaps the most natural way to introduce complex numbers is through real quadratic equations; equations of the form

$$ax^2 + bx + c = 0, \tag{1.1}$$

where a, b, and c are real numbers. It is well known that when the discriminant $b^2 - 4ac$ is positive, this equation has two real distinct roots given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1.2}$$

and when $b^2 - 4ac = 0$, there is a repeated real root $x = -b/(2a)$. But when $b^2 - 4ac < 0$, there are no real numbers that satisfy the equation.

Complex numbers enable us to solve quadratic equations with negative discriminants. The most fundamental complex number is denoted by i; it is defined as a number whose square is -1,

$$i^2 = -1. \tag{1.3}$$

In other words, i is a solution of the quadratic equation

$$z^2 + 1 = 0 \tag{1.4}$$

(with discriminant equal to -4).

The introduction of i is often met with apprehension because it does not conform to our past experience with real numbers. But the transition from real numbers to complex numbers is completely analogous to the transition from rational numbers to real numbers. All rational numbers satisfy linear equations

$$ax + b = 0,$$

with integer coefficients a and b, where $a \neq 0$. For example, $3/2$ satisfies $2x - 3 = 0$. The quadratic equation $x^2 - 3 = 0$, on the other hand, does not have rational solutions. We denote the solutions of $x^2 - 3 = 0$ by $\pm\sqrt{3}$. But what is $\sqrt{3}$? It is not a rational number; it cannot be expressed as an integer divided by an integer, and it does not have a repeating decimal representation. We usually say that $\sqrt{3}$ is a number that multiplies itself to give 3. Over the years, you have accepted this explanation and now regard $\sqrt{3}$ with no trepidation whatsoever. The same is true when we extend the real number system to the complex number system. Equation 1.4 has no real solutions. We denote a solution of the equation by the letter i. But what is i? It is a number that, when multiplied by itself, gives -1. As we see then, the definition of i is much the same as the definition of $\sqrt{3}$, and with a little experience, you will regard i with no more apprehension that $\sqrt{3}$.

EXERCISES 1.1

In these exercises we discuss operations on real numbers of the form $a + b\sqrt{3}$, where a and b are rational numbers. Each operation is analogous to an operation on complex numbers in Section 1.2.

1. Show that $a + b\sqrt{3} = c + d\sqrt{3}$ if and only if $a = c$ and $b = d$.

2. How do we add $a + b\sqrt{3}$ and $c + d\sqrt{3}$?

3. How do we subtract $c + d\sqrt{3}$ from $a + b\sqrt{3}$?

4. How do we multiply $a + b\sqrt{3}$ and $c + d\sqrt{3}$?

5. How do we divide $a + b\sqrt{3}$ by $c + d\sqrt{3}$?

§1.2 Complex Numbers

The fundamental complex number is i, a number whose square is -1. The complete set of complex numbers is introduced in the following definition.

Definition 1.1 The **complex number system** $\mathcal{C}$ consists of all "numbers" of the form

$$z = x + yi, \tag{1.5}$$

where x and y are real, and i satisfies $i^2 = -1$.

Examples are $4+2i$, $3-2i$, $-3+\pi i$, and $-6-3i$. We liken $x+yi$ to numbers of the form $a+b\sqrt{3}$ in Exercises 1.1; corresponding to each of the complex numbers in the previous sentence, we would have $4+2\sqrt{3}$, $3-2\sqrt{3}$, $-3+\pi\sqrt{3}$, and $-6-3\sqrt{3}$. The number $3-2\sqrt{3}$ cannot be written exactly in a simpler form, the 3 and -2 cannot in any way be combined. Likewise, the 3 and -2 in the complex number $3-2i$ are distinct parts, so much so that we give them special names. The number x in equation 1.5 is called the **real part** of the complex number z; it is denoted by $x = \operatorname{Re} z$. The number y is called the **imaginary part** of z, denoted by $y = \operatorname{Im} z$. For example, $\operatorname{Re}(3-2i) = 3$ and $\operatorname{Im}(3-2i) = -2$. Both the real and imaginary parts of a complex number are themselves real numbers. The real number system is a subset of $\mathcal{C}$ obtained when $y = 0$.

To appreciate many results involving complex numbers, it is helpful to have geometric visualizations. Because a complex number has two distinct parts, its real and imaginary parts, and they are independent, a two-dimensional model is required to specify $\mathcal{C}$. We choose a plane, called the **complex** (Argand) **plane**. In particular, some fixed point O is chosen to represent the complex number $0+0i$. Through O are drawn two mutually perpendicular axes (Figure 1.1), one called the **real axis**, and the other called the **imaginary axis**. The complex number $x+yi$ is then represented by the point x units in the real direction and y units in the imaginary direction. For example, the complex numbers $1+2i$, $-1-i$, $4-3i$, and $-2+2i$ are shown in Figure 1.2. With this geometric representation, there is a one-to-one correspondence between numbers in $\mathcal{C}$ and points in the complex plane. In addition, the real number system (which is a subset of $\mathcal{C}$) is represented by points on the real axis. Complex numbers of the form $z = yi$ are said to be **purely imaginary**. They lie on the imaginary axis.

The point in the complex plane representing the number $x+yi$ is the same as the point (x, y) in the Cartesian plane, and we therefore call $x+yi$ the **Cartesian form** for a complex number.

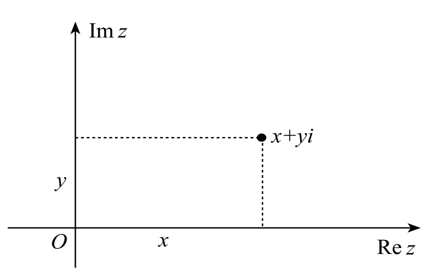


Figure 1.1

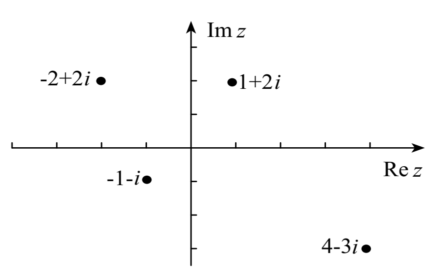


Figure 1.2

In order to add, subtract, multiply, and divide complex numbers, we first define what it means for two complex numbers to be equal, and we do so by analogy with Exercise 1 in Section 1.1.

Definition 1.2 Two complex numbers $z_1 = x + yi$ and $z_2 = a + bi$ are said to be equal if their real parts are equal and their imaginary parts are equal; that is,

$$x + yi = a + bi \qquad \Longleftrightarrow \qquad x = a \text{ and } y = b. \tag{1.6}$$

Geometrically, two complex numbers are equal if they correspond to the same point in the complex plane. Do not pass this definition off too lightly; we use it many times in our work. It allows us to replace an equation with complex numbers by a pair of equations with real numbers.

We now learn how to add, subtract, multiply, and divide complex numbers. Addition and subtraction of complex numbers are defined as follows.

Definition 1.3 If $z_1 = x + yi$ and $z_2 = a + bi$, then

$$z_1 + z_2 = (x + a) + (y + b)i, \tag{1.7a}$$
$$z_1 - z_2 = (x - a) + (y - b)i. \tag{1.7b}$$

In words, complex numbers are added and subtracted by adding and subtracting their real parts and adding and subtracting their imaginary parts. (Compare this with the addition and subtraction of numbers of the form $a + b\sqrt{3}$ in Exercises 2 and 3 of Section 1.1.) For example,

$$(3 - 2i) + (6 + i) = (3 + 6) + (-2 + 1)i = 9 - i,$$

$$(3 - 2i) - (6 + i) = (3 - 6) + (-2 - 1)i = -3 - 3i.$$

It is sometimes convenient to consider a complex number z to be represented geometrically by the vector joining O to the point representing z (Figure 1.3), rather than the point itself. The real and imaginary parts of the complex number correspond to the x- and y-components of the vector. It is clear that law 1.7a for addition of complex numbers is the same as the law of addition of vectors. Thus, to add two complex numbers z_1 and z_2 geometrically, we perform vector addition by means of either a triangle (Figure 1.4a) or a parallelogram (Figure 1.4b) on their geometric representations as vectors.

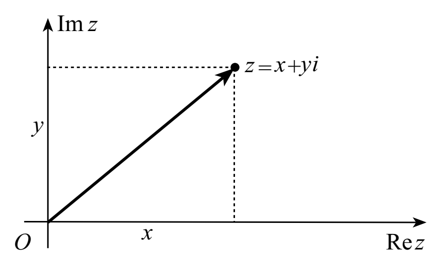


Figure 1.3

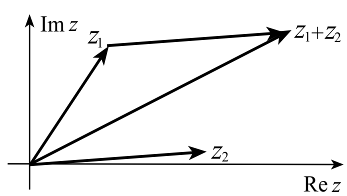


Figure 1.4a

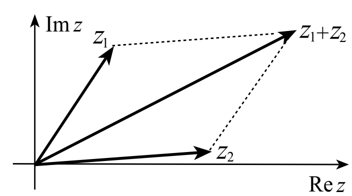


Figure 1.4b

Complex numbers can be subtracted vectorially also. Subtraction by triangles is shown in Figure 1.5a and by parallelograms in Figure 1.5b.

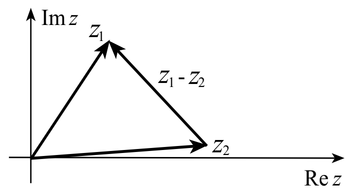


Figure 1.5a

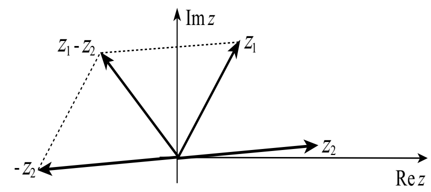


Figure 1.5b

Complex numbers are multiplied according to the following definition.

Definition 1.4 If $z_1 = x + yi$ and $z_2 = a + bi$, then

$$z_1 z_2 = (x + yi)(a + bi) = (xa - yb) + (xb + ya)i. \tag{1.8}$$

For example,

$$(3 - 2i)(6 + i) = [(3)(6) - (-2)(1)] + [(3)(1) + (-2)(6)]i = (18 + 2) + (3 - 12)i = 20 - 9i.$$

It is not necessary to memorize equation 1.8 when we note that this definition is precisely what we would expect if the usual laws for multiplying binomials were applied, together with the fact that $i^2 = -1$:

$$\begin{aligned}(3 - 2i)(6 + i) &= (3)(6) + (3)(i) + (-2i)(6) + (-2i)(i)\\ &= 18 + 3i - 12i - 2i^2\\ &= 18 - 9i - 2(-1)\\ &= 20 - 9i.\end{aligned}$$

With addition, subtraction, and multiplication taken care of, it is natural to turn to division of complex numbers. If we accept that division of any complex number by itself should be equal to 1, and that ordinary rules of algebra should prevail, a definition of division of complex numbers is not necessary. It follows from equation 1.8. When $z_1 = x + yi$ and $z_2 = a + bi$, we calculate

$$\frac{z_1}{z_2} = \frac{x + yi}{a + bi}$$

by multiplying numerator and denominator by $a - bi$. This results in

$$\begin{aligned}\frac{z_1}{z_2} = \frac{x + yi}{a + bi} = \frac{(x + yi)(a - bi)}{(a + bi)(a - bi)} &= \frac{(xa + yb) + (-xb + ya)i}{a^2 + b^2} \qquad \text{(using 1.8)}\\ &= \left(\frac{xa + yb}{a^2 + b^2}\right) + \left(\frac{ya - xb}{a^2 + b^2}\right) i. \qquad (1.9)\end{aligned}$$

For example,

$$\frac{3 - 2i}{6 + i} = \frac{(3 - 2i)(6 - i)}{(6 + i)(6 - i)} = \frac{16 - 15i}{37} = \frac{16}{37} - \frac{15}{37}i.$$

In summary, addition, subtraction, multiplication, and division of complex numbers are performed using ordinary rules of algebra with the extra condition that i^2 is always replaced by -1.

Example 1.1 Write the following complex numbers in Cartesian form:

$$\text{(a)}\quad (3+i)(2-i)^2 - i \qquad \text{(b)}\quad \frac{i^3}{2+i} \qquad \text{(c)}\quad \frac{4-3i^2+2i}{(2-2i^{15})^2}$$

Solution (a) $(3+i)(2-i)^2 - i = (3+i)(3-4i) - i = (13-9i) - i = 13-10i$

(b) $\dfrac{i^3}{2+i} = \dfrac{-i(2-i)}{(2+i)(2-i)} = \dfrac{-1-2i}{5} = -\dfrac{1}{5} - \dfrac{2}{5}i$

(c) $\dfrac{4-3i^2+2i}{(2-2i^{15})^2} = \dfrac{4+3+2i}{(2+2i)^2} = \dfrac{7+2i}{8i} = \dfrac{(7+2i)(-i)}{(8i)(-i)} = \dfrac{2-7i}{8} = \dfrac{1}{4} - \dfrac{7}{8}i$ •

Notice in part (c) of this example that we multiplied numerator and denominator by $-i$ rather than $-8i$; the result is the same in either case. Both lead to a real denominator.

Definition 1.5 The **complex conjugate** $\overline{z}$ of a complex number $z = x + yi$ is

$$\overline{z} = x - yi. \tag{1.10}$$

For example, the complex conjugates of $1+2i$ and $3-4i$ are $\overline{1+2i} = 1-2i$ and $\overline{3-4i} = 3+4i$. Geometrically, $\overline{z}$ is the reflection of z in the real axis (Figure 1.6).

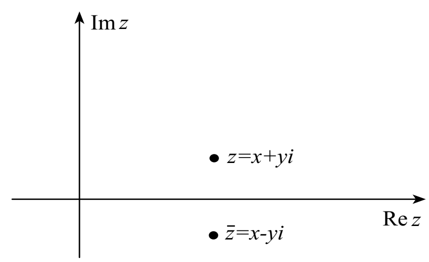


Figure 1.6

The procedure by which two complex numbers are divided (equation 1.9) can be stated as follows. To divide z_1 by z_2, multiply z_1 and z_2 by $\overline{z_2}$,

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}}.$$

The denominator will be real, and the Cartesian form is immediate.

Let us now return to the discussion of quadratic equation 1.1. When the discriminant is positive, the equation has two real solutions, and when the discriminant is zero, we regard the quadratic as having two real solutions which are identical. A solution of

$$x^2 + 1 = 0,$$

(which has a negative discriminant) is the complex number i, but so also is $-i$ because $(-i)^2 + 1 = -1 + 1 = 0$. The quadratic equation

$$x^2 + 16 = 0$$

has two solutions $x = \pm 4i$. If we apply quadratic formula 1.2 to the equation

$$x^2 + 2x + 5 = 0,$$

the result is

$$x = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}.$$

By $\sqrt{-16}$ we would seem to mean the number which multiplied by itself is -16. But there are two such complex numbers, namely $\pm 4i$. We make the agreement that $\sqrt{-16}$ shall denote that complex number whose square is -16, and that has a positive imaginary part. By this agreement,

$$\sqrt{-16} = 4i \qquad \text{and} \qquad -\sqrt{-16} = -4i.$$

The quadratic formula applied to $x^2 + 2x + 5 = 0$ therefore gives the two complex numbers

$$x = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

It is straightforward to verify that these two complex conjugates do indeed satisfy the quadratic equation $x^2 + 2x + 5 = 0$.

The agreement made in this last example is worth reiterating as a general principle:

When $a > 0$ (is a real number),

$$\sqrt{-a} = \sqrt{a}\,i. \tag{1.11}$$

We call $\sqrt{a}\,i$ the **principal square root** of $-a$; the other square root is $-\sqrt{a}\,i$.

The above examples lead us to the following result.

Theorem 1.1 Every real quadratic equation

$$ax^2 + bx + c = 0 \tag{1.12a}$$

has two solutions. They are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.12b}$$

When $b^2 - 4ac > 0$, the roots are real and distinct; when $b^2 - 4ac = 0$, the roots are real and equal; and when $b^2 - 4ac < 0$, the roots are complex conjugates.

That formula 1.12b defines two complex numbers that satisfy equation 1.12a, even when $b^2 - 4ac < 0$, is easily verified by substituting from 1.12b into 1.12a.

Example 1.2 Find all solutions of the following equations:

(a) $x^2 + x + 3 = 0$ (b) $x^2 - 6x + 9 = 0$ (c) $2x^2 + 17x - 2 = 0$ (d) $x^4 + 5x^2 + 4 = 0$

Solution (a) By quadratic formula 1.12b,

$$x = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}\,i.$$

(b) This quadratic can be factored, $0 = x^2 - 6x + 9 = (x - 3)^2$, and therefore has a double root $x = 3$.
(c) By formula 1.12b,

$$x = \frac{-17 \pm \sqrt{289 + 16}}{4} = \frac{-17 \pm \sqrt{305}}{4},$$

two real roots.

(d) If we set $y = x^2$, then

$$0 = x^4 + 5x^2 + 4 = y^2 + 5y + 4 = (y + 4)(y + 1).$$

Consequently, y is equal to -1 or -4. Since $y = x^2$, we set $x^2 = -1$ and $x^2 = -4$. These equations have roots $x = \pm i$ and $x = \pm 2i$.•

Theorem 1.1 is valid even when coefficients a, b, and c in equation 1.12a are complex, in which case we speak of a complex quadratic equation. However, formula 1.12b may lead to square roots of complex numbers when a, b, and c are complex. This is not a problem; using equation 1.6 for equality of complex numbers we can find the square roots of any complex number. (Other, and better, methods will be developed in Sections 1.5 and 3.6.)

Finding the square roots of a complex number Z is equivalent to solving the following equation for z,

$$z^2 = Z. \tag{1.13}$$

If X and Y are the real and imaginary parts of Z ($Z = X + Yi$), and we let x and y be the real and imaginary parts of z, then

$$X + Yi = (x + yi)^2 = x^2 - y^2 + 2xyi.$$

But according to equation 1.6, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal; that is, we may set

$$x^2 - y^2 = X, \qquad 2xy = Y.$$

We now have two real equations in two real unknowns x and y, the real and imaginary parts of z. Once they are solved, and there should be two solutions, we have the two square roots of Z. We illustrate by finding the square roots of i.

If we set $z = x + yi$ in the equation $z^2 = i$, the result is

$$i = (x + yi)^2 = (x^2 - y^2) + 2xyi.$$

Equating real and imaginary parts of the complex numbers on each side of this equation gives

$$x^2 - y^2 = 0, \qquad 2xy = 1.$$

From the second of these, $y = 1/(2x)$, which we substitute into the first,

$$0 = x^2 - \frac{1}{(2x)^2} = \frac{4x^4 - 1}{4x^2}.$$

This implies that

$$0 = 4x^4 - 1 = (2x^2 + 1)(2x^2 - 1).$$

Because x must be real, we have two solutions $x = \pm 1/\sqrt{2}$; and correspondingly, $y = \pm 1/\sqrt{2}$. Thus, the square roots of i are

$$z = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right).$$

One of these would be denoted by $\sqrt{i}$ and the other by $-\sqrt{i}$. The following agreement extends equation 1.11 to all complex numbers:

Definition 1.6 The **principal square root** of $a+bi$, denoted by $\sqrt{a+bi}$, is the square root with nonnegative real part.

With this agreement,

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i \qquad \text{and} \qquad -\sqrt{i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i.$$

We have uncovered here a most important principle. A complex equation can be replaced by two real equations, the equations obtained by equating real and imaginary parts of each side of the complex equation. Watch for this principle; we use it often.

Unfortunately the above technique for finding square roots of complex numbers does not generalize easily to finding cube roots, fourth roots, etc. For example, consider finding the cube roots of $1-2i$. This is equivalent to solving the equation $z^3 = 1-2i$, and if we set $z = x+yi$,

$$1-2i = (x+yi)^3 = (x^3 - 3xy^2) + (3x^2y - y^3)i.$$

When we equate real and imaginary parts, we obtain the unattractive, real equations

$$x^3 - 3xy^2 = 1, \quad 3x^2y - y^3 = -2.$$

But, as we mentioned earlier, Sections 1.5 and 3.6 develop easier methods for finding roots of complex numbers of all orders.

With the ability to find square roots of complex numbers, we can use formula 1.12b to find roots of complex quadratic equations. We illustrate in the following example.

Example 1.3 Find roots of the complex quadratic equation

$$iz^2 + 2z + 3 = 0.$$

Solution Quadratic formula 1.12b gives

$$z = \frac{-2 \pm \sqrt{4-12i}}{2i} = \frac{1}{i}\left(-1 \pm \sqrt{1-3i}\right) = -i\left(-1 \pm \sqrt{1-3i}\right).$$

To find the square roots of $1-3i$, we set $w = x+yi$ in $w^2 = 1-3i$,

$$1-3i = (x+yi)^2 = (x^2-y^2) + 2xyi.$$

Equating real and imaginary parts gives

$$x^2 - y^2 = 1, \qquad 2xy = -3.$$

From the second of these, $y = -3/(2x)$, which we substitute into the first,

$$x^2 - \left(\frac{-3}{2x}\right)^2 = 1 \qquad \Longrightarrow \qquad 4x^4 - 4x^2 - 9 = 0.$$

This is essentially a quadratic equation in x^2, namely, $4(x^2)^2 - 4(x^2) - 9 = 0$, and therefore

$$x^2 = \frac{4 \pm \sqrt{16+144}}{8} = \frac{1 \pm \sqrt{10}}{2}.$$

Since x^2 must be positive, it follows that

$$x^2 = \frac{1+\sqrt{10}}{2} \qquad \Longrightarrow \qquad x = \pm\sqrt{\frac{1+\sqrt{10}}{2}}.$$

Correspondingly,

$$y = \frac{\mp 3}{\sqrt{2+2\sqrt{10}}}.$$

Hence, the square roots of $1-3i$ are

$$\sqrt{1-3i} = \sqrt{\frac{1+\sqrt{10}}{2}} - \frac{3}{\sqrt{2+2\sqrt{10}}}\,i, \qquad -\sqrt{1-3i} = -\sqrt{\frac{1+\sqrt{10}}{2}} + \frac{3}{\sqrt{2+2\sqrt{10}}}\,i,$$

and the solutions of the quadratic equation $iz^2+2z+3=0$ are

$$z = -i\left[-1 \pm \left(\sqrt{\frac{1+\sqrt{10}}{2}} - \frac{3}{\sqrt{2+2\sqrt{10}}}\,i\right)\right] = \frac{\pm 3}{\sqrt{2+2\sqrt{10}}} + \left(1 \pm \sqrt{\frac{1+\sqrt{10}}{2}}\right) i.\bullet$$

EXERCISES 1.2

1. Show each of the following complex numbers in the complex plane: (a) $2-i$, (b) $3+4i$, (c) $-1-5i$, (d) $-3+2i$, (e) $5i$, and (f) $2(1+i)$

In Exercises 2–15 write the complex expression in Cartesian form.

2. $(2+4i)-(3-2i)$

3. $(1+2i)^2$

4. $(-2+i)(3-4i)$

5. $3i(4i-1)^2$

6. i^3-3i^2+2i+4

7. $(1+i)^6$

8. $\dfrac{1-i}{3+2i}$

9. $\dfrac{(3+i)^2}{2-i}$

10. $i^{24}-3i^{13}+4$

11. $(i-2)[(2+i)(1-i)+3i-2]$

12. $6i\left(\dfrac{1+i}{2-i}\right)+3\left(\dfrac{i-4}{2i+1}\right)$

13. $\overline{2+i}-(3+\overline{4i})$

14. $\overline{1+i}^{\,2}+\overline{(1+i)^2}$

15. $\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\,i\right)^3$

16. Verify the following properties for the complex conjugation operation:
(a) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$
(b) $\overline{z_1-z_2} = \overline{z_1}-\overline{z_2}$
(c) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
(d) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$
(e) $\overline{z^n} = \overline{z}^{\,n}$, n a positive integer

17. (a) If $P(z)$ is a polynomial with real coefficients, use the results of Exercise 16 to verify that $\overline{P(z)} = P(\overline{z})$.

(b) Verify that if z is a root of a polynomial equation $P(z) = 0$, with real coefficients, then so also is $\overline{z}$. In other words, complex roots of a real polynomial equation always occur in complex conjugate pairs.

18. Verify that all complex numbers z satisfying the equation $z\overline{z} = r^2$, $r > 0$ a real constant, lie on a circle. What are its centre and radius?

19. Prove that if $z_1 z_2 = 0$, then at least one of z_1 and z_2 must be zero.

20. We have made the agreement that when $a > 0$ is a real number, $\sqrt{-a}$ denotes a complex number with positive imaginary part (see equation 1.11). Show that with this agreement,

$$\sqrt{z_1 z_2} \text{ is not always equal to } \sqrt{z_1}\sqrt{z_2}.$$

Hint: Try $z_1 = z_2 = -1$.

21. Explain the fallacy in

$$-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

In Exercises 22–31 find all solutions of the equation.

22. $x^2 + 5x + 3 = 0$

23. $x^2 + 3x + 5 = 0$

24. $x^2 + 8x + 16 = 0$

25. $x^2 + 2x - 7 = 0$

26. $x^2 + 2x + 7 = 0$

27. $4x^2 - 2x + 5 = 0$

28. $\sqrt{3}x^2 + 5x + \sqrt{15} = 0$

29. $x^4 + 4x^2 - 5 = 0$

30. $x^4 + 4x^2 + 3 = 0$

31. $x^4 + 6x^2 + 3 = 0$

32. Find two numbers whose sum is 6 and whose product is 10.

In Exercises 33–36 find square roots of the complex number.

33. $-7 - 24i$

34. $21 - 20i$

35. $2 + i$

36. $-1 + i$

In Exercises 37–42 use quadratic formula 1.12b to find both solutions of the complex quadratic equation.

37. $z^2 + (2 - i)z + 1 - i = 0$

38. $2iz^2 + 4z + 3 - 2i = 0$

39. $z^2 + 2z + i = 0$

40. $iz^2 + 2iz + 3 = 0$

41. $z^2 + 4iz - 1 + i = 0$

42. $z^2 + (2 - i)z + 1 + i = 0$

43. Show that the solutions of the equation in Exercise 37 can be obtained by letting $z = x + yi$, substituting into the equation, and equating real and imaginary parts.

44. Repeat the procedure of Exercise 43 on the equation in Exercise 38.

45. Show that the line joining the complex numbers z_1 and z_2 is perpendicular to the line joining z_3 and z_4 if and only if $\text{Re}\,[(z_1 - z_2)(\overline{z_3} - \overline{z_4})] = 0$.

46. Show that the line joining the complex numbers z_1 and z_2 is parallel to the line joining z_3 and z_4 if and only if $\text{Im}\,[(z_1 - z_2)(\overline{z_3} - \overline{z_4})] = 0$.

§1.3 Polar Representation of Complex Numbers

In Section 1.2 we worked with complex numbers in Cartesian form, and this proved sufficient to solve quadratic equations. For other applications, it is often advantageous to have what is called the *polar representation* of a complex number. Even more useful is the *exponential representation* of a complex number; it is developed in Section 1.4. To find the polar representation, we define the modulus and argument of a complex number.

Definition 1.7 The **modulus** of a complex number $z = x + yi$ is

$$r = |z| = \sqrt{x^2 + y^2}. \tag{1.14}$$

It is the length of the line segment joining the points in the complex plane representing the complex numbers $z = 0$ and $z = x + yi$ (Figure 1.7). It is unique; a complex number has exactly one modulus. For example, $|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$ and $|-3+4i| = \sqrt{(-3)^2+4^2} = 5$. Notice that when z is real, say $z = x$, then $|z| = \sqrt{x^2} = |x|$, where $|x|$ is the absolute value of x. In other words, when z is real, the modulus bars are absolute value bars. Often useful is the fact that

$$|z|^2 = z\overline{z}. \tag{1.15}$$

Definition 1.8 An **argument** of a nonzero complex number $z = x + yi$, usually denoted by θ or $\arg z$, is an angle of rotation of the positive real axis in a counterclockwise direction to the line segment joining $z = 0$ and $z = x + yi$. Clockwise rotations are regarded as negative.

Notice that in this definition we said "an" argument rather than "the" argument of a complex number. The reason for this is that arguments of complex numbers are not unique. For instance, one possible value for an argument of the complex number $z = 1 - i$ (Figure 1.8) is $\theta = 7\pi/4$, but another is $\theta = -\pi/4$. In fact, there are infinitely many possibilities for θ, namely, $\theta = 2k\pi - \pi/4$, where k is any integer. We call that value of the argument of a nonzero complex number z which satisfies the restriction $-\pi < \theta \leq \pi$ the **principal value** of the argument, and denote it with a capital A,

$$-\pi < \operatorname{Arg} z \leq \pi. \tag{1.16}$$

For $z = 1 - i$, the principal value of the argument is $-\pi/4$.

We have a choice to make for an argument of $z = 0$. We could accept that any real number could be used as an argument for $z = 0$, and any number in the interval $(-\pi, \pi]$ could be used as a principal value for its argument. On the other hand, since we eventually wish to regard $\arg z$ and $\operatorname{Arg} z$ as functions, which must therefore be single-valued, we choose instead not to assign an argument to $z = 0$.

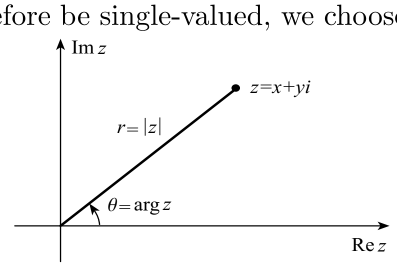


Figure 1.7

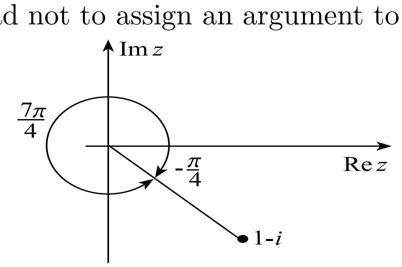


Figure 1.8

The real and imaginary parts of a complex number can be expressed in terms of its modulus and argument. The triangle in Figure 1.9 indicates that

$$x = r\cos\theta, \qquad y = r\sin\theta. \qquad (1.17)$$

It is easy to reverse these equations and express r in terms of x and y; equation 1.14 does this. But the case for θ is more complicated. Certainly we can write that

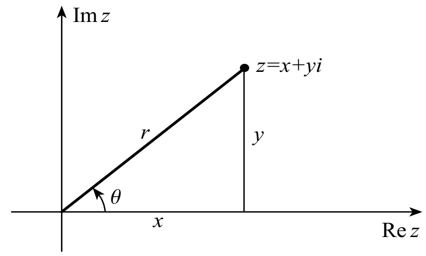


Figure 1.9

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad \text{and} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \qquad (1.18)$$

and these equations do indeed define all arguments of $z = x + yi$. In an attempt to find a single equation defining θ in terms of x and y, there is a tendency to divide the second equation by the first

$$\tan\theta = \frac{y}{x},$$

and take this one step further to

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right). \qquad (1.19)$$

Unfortunately, principal values† of the inverse tangent function are between $\pm\pi/2$, and therefore this equation gives a correct argument only for complex numbers in the first and fourth quadrants.

How then are we to find arguments for given $z = x + yi$? Certainly we can find all angles satisfying equation 1.18. Alternatively, we can find the angle θ defined by equation 1.19. If z is in the first or fourth quadrants ($x > 0$), then all possible arguments are $\theta + 2k\pi$, where k is an integer. When z is in the second quadrant, arguments are $(\theta + \pi) + 2k\pi = \theta + (2k+1)\pi$; and when z is in the third quadrant, arguments are $(\theta - \pi) + 2k\pi = \theta + (2k-1)\pi$.

Example 1.4 Find all arguments and the principal value of the argument for the following complex numbers:

(a) $1 + \sqrt{3}i$ (b) $2 - 2i$ (c) $-4 + 3i$ (d) $-3 - 2i$

Solution We have shown all four complex numbers in Figure 1.10.
(a) Since $\mathrm{Tan}^{-1}(\sqrt{3}/1) = \pi/3$, arguments of $1 + \sqrt{3}i$ are $\pi/3 + 2k\pi = (6k+1)\pi/3$. The principal value of the argument is $\pi/3$.

(b) One argument of $2 - 2i$ is clearly $-\pi/4$, and therefore all arguments are $2k\pi - \pi/4 = (8k-1)\pi/4$. The principal value is $-\pi/4$.

† Capital letters on inverse trigonometric functions denote principal values. For the inverse sine, cosine, and tangent functions, capital letters denote the principal values $-\pi/2 \le \mathrm{Sin}^{-1}a \le \pi/2$, $0 \le \mathrm{Cos}^{-1}a \le \pi$, and $-\pi/2 < \mathrm{Tan}^{-1}a < \pi/2$.

(c) Since $\mathrm{Tan}^{-1}[3/(-4)] \approx -0.6435$, and $-4+3i$ is in the second quadrant, it follows that arguments are $(\pi - 0.6435) + 2k\pi = (2k+1)\pi - 0.6435$. The principal value is $\pi - 0.6435$.

(d) With $\mathrm{Tan}^{-1}[-2/(-3)] \approx 0.5880$ and $-3-2i$ in the third quadrant, arguments are $(0.5880 - \pi) + 2k\pi = (2k-1)\pi + 0.5880$. The principal value is $0.5880 - \pi$.•

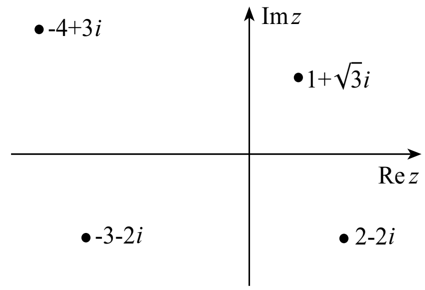


Figure 1.10

If expressions 1.17 are substituted into the Cartesian form $z = x + yi$ for a complex number,

$$z = x + yi = r\cos\theta + (r\sin\theta)i.$$

When r is factored from both terms,

$$z = r(\cos\theta + \sin\theta\, i). \tag{1.20}$$

This expression is called the **polar representation** of a complex number. Realize again that it is not unique. If 1.20 is a polar representation of a complex number, then

$$r[\cos(\theta + 2k\pi) + \sin(\theta + 2k\pi)\, i],$$

for any integer k, is also a polar representation for the same complex number.

Example 1.5 Find polar representations for $-1+i$ and $2-3i$.

Solution Since the modulus of $-1+i$ is $\sqrt{2}$, and an argument is $3\pi/4$ (Figure 1.11), a polar representation is

$-1 + i = \sqrt{2}[\cos(3\pi/4) + \sin(3\pi/4)\, i]$.

The modulus of $2-3i$ is $\sqrt{13}$, and an argument is $\mathrm{Tan}^{-1}(-3/2) \approx -0.983$. Consequently,

$$\begin{aligned} 2 - 3i &= \sqrt{13}\{\cos[\mathrm{Tan}^{-1}(-3/2)] \\ &\quad + \sin[\mathrm{Tan}^{-1}(-3/2)]\, i\} \\ &\approx \sqrt{13}[\cos(-0.983) + \sin(-0.983)\, i]. \bullet \end{aligned}$$

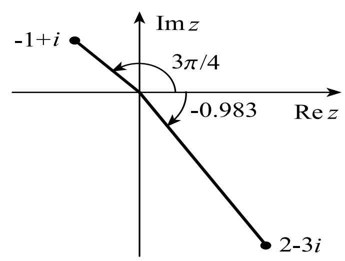


Figure 1.11

Example 1.6 Is $-2[\cos(\pi/4) + \sin(\pi/4)\, i]$ the polar form for a complex number?

Solution No, because -2 cannot be the modulus of a complex number (moduli must be nonnegative). To express this complex number in polar form, we write

$$-2[\cos(\pi/4) + \sin(\pi/4)\, i] = 2[-\cos(\pi/4) - \sin(\pi/4)\, i] = 2[\cos(5\pi/4) + \sin(5\pi/4)\, i]. \bullet$$

Two complex numbers are equal if and only if they have the same real parts and the same imaginary parts. How do we phrase equality in terms of moduli and arguments? Certainly complex numbers can be equal only if they have the same moduli. Their arguments need not be the same, however, but they must differ by a multiple of 2π. In other words, we have:

Two nonzero complex numbers z_1 and z_2 are equal if and only if

$$|z_1| = |z_2|, \tag{1.21a}$$

and

$$\arg z_1 = \arg z_2 + 2k\pi, \;\; k \text{ an integer.} \tag{1.21b}$$

Since principal values of arguments of complex numbers are unique, an alternative to conditions 1.21 is that nonzero complex numbers z_1 and z_2 are equal if and only if

$$|z_1| = |z_2|, \tag{1.22a}$$

and

$$\operatorname{Arg} z_1 = \operatorname{Arg} z_2. \tag{1.22b}$$

Complex numbers are easily added and subtracted in Cartesian form. Even multiplication and division of two complex numbers is relatively easy in Cartesian form. On the other hand, multiplication and division in polar form lead to properties that are not evident in Cartesian form. For example, if $z_1 = r_1(\cos\theta_1 + \sin\theta_1\, i)$ and $z_2 = r_2(\cos\theta_2 + \sin\theta_2\, i)$ are any two complex numbers, their product is

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + \sin\theta_1\, i)][r_2(\cos\theta_2 + \sin\theta_2\, i)] \\ &= r_1 r_2[(\cos\theta_1\, \cos\theta_2 - \sin\theta_1\, \sin\theta_2) + (\sin\theta_1\, \cos\theta_2 + \sin\theta_2\, \cos\theta_1)i] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\, i]. \end{aligned} \tag{1.23}$$

Because the last expression is the polar representation of $z_1 z_2$, we have proved the following facts:

1. The modulus of the product of two complex numbers is the product of their moduli; that is,

$$|z_1 z_2| = |z_1||z_2|. \tag{1.24a}$$

2. An argument of the product of two complex numbers is the sum of their arguments; that is,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{1.24b}$$

Be wary of how you interpret 1.24b; it is not an equation in the normal sense of equation. Each of the three terms has infinitely many possible values (since the argument of a complex number is determined only to multiples of 2π). What this "equation" says is that given any two terms in 1.24b, there is a value of the third that makes 1.24b an equation. For example, positive arguments of $z_1 = 1 + i$ and $z_2 = 1 - i$ are $\pi/4$ and $7\pi/4$. The product of these complex numbers is $z_1 z_2 = (1+i)(1-i) = 2$. An argument for the complex number 2 is 0, but certainly the arguments $\arg z_1 = \pi/4$, $\arg z_2 = 7\pi/4$, and $\arg z_1 z_2 = 0$ do not satisfy 1.24b. On the other hand, the arguments $\arg z_1 = \pi/4$, $\arg z_2 = 7\pi/4$, and $\arg z_1 z_2 = 2\pi$ do satisfy 1.24b.

Condition 1.24b cannot be replaced by principal values, because, in general, $\operatorname{Arg} z_1 z_2$ may not be equal to $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$. This is the situation when $z_1 = z_2 = -1$. In this case, $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} 1 = 0$, and $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \pi + \pi = 2\pi$.

A similar analysis shows that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\, i]. \tag{1.25}$$

In other words, we have the following properties corresponding to equations 1.24:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{1.26a}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{1.26b}$$

Example 1.7 Find $z_1 z_2$ and z_1/z_2 if $z_1 = 6(\cos 1.1 + \sin 1.1\, i)$ and $z_2 = 3(\cos 2.4 + \sin 2.4\, i)$.

Solution According to equations 1.23 and 1.25,

$$z_1 z_2 = 18(\cos 3.5 + \sin 3.5\, i) \qquad \text{and} \qquad \frac{z_1}{z_2} = 2[\cos(-1.3) + \sin(-1.3)\, i].\bullet$$

The polar form for a complex number can be very useful when we raise a complex number to a power, say $(1+\sqrt{3}i)^8$. To do this we derive a simple formula for z^n when n is a positive integer. With $z = r(\cos\theta + \sin\theta\, i)$,

$$z^2 = [r(\cos\theta + \sin\theta\, i)][r(\cos\theta + \sin\theta\, i)] = r^2(\cos 2\theta + \sin 2\theta\, i);$$

$$z^3 = zz^2 = [r(\cos\theta + \sin\theta\, i)][r^2(\cos 2\theta + \sin 2\theta\, i)] = r^3(\cos 3\theta + \sin 3\theta\, i).$$

The pattern emerging (which could be proved by mathematical induction) is

$$z^n = r^n[\cos(n\theta) + \sin(n\theta)\, i]. \tag{1.27}$$

This result is often called **DeMoivre's** theorem.

Example 1.8 Evaluate $(1+\sqrt{3}i)^8$.

Solution With $|1+\sqrt{3}i| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$, and $\text{Arg}\,(1+\sqrt{3}i) = \text{Tan}^{-1}(\sqrt{3}) = \pi/3$,

$$\begin{aligned}(1+\sqrt{3}i)^8 &= \{2[\cos(\pi/3) + \sin(\pi/3)\, i]\}^8 = 2^8[\cos(8\pi/3) + \sin(8\pi/3)\, i]\\ &= 256[\cos(2\pi/3) + \sin(2\pi/3)\, i] = 256\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\, i\right) = 128(-1+\sqrt{3}i).\bullet\end{aligned}$$

DeMoivre's theorem can also be used to find roots of complex numbers; that is, find the square roots of $1+i$, the cube roots of $-2+3i$, the fourth roots of $6-2i$, etc. In general, we can find the n n^{th} roots of any given complex number Z. What we are really doing when we find these roots is solving the equation

$$z^n = Z, \tag{1.28}$$

where $n > 1$ is an integer, and Z is a given complex number. As we said, we can use DeMoivre's theorem to find all solutions of 1.28, but there is an easier way. We use what is called the *exponential form* of complex numbers (Section 1.4). It is equivalent to the polar form, using modulus r and argument θ, but it is so much simpler. In fact, we regard the polar form of a complex number as a stepping stone to the preferred exponential form, and only use Cartesian and exponential forms.

EXERCISES 1.3

In Exercises 1–6 express the complex number in polar form.

1. $-1+i$

2. $-2i$

3. $\sqrt{3}-i$

4. $3+4i$

5. $-1-2i$

6. $-2[\cos(\pi/3)-\sin(\pi/3)\,i]$

In Exercises 7–10 express the complex number in Cartesian form.

7. $3[\cos(\pi/6)+\sin(\pi/6)\,i]$

8. $\cos(1.4)+\sin(1.4)\,i$

9. $2[\cos(\pi/4)+\sin(\pi/4)\,i]^2$

10. $\dfrac{4[\cos(2\pi/3)+\sin(2\pi/3)\,i]}{\cos(\pi/6)+\sin(\pi/6)\,i}$

In Exercises 11–16 simplify the expression as much as possible.

11. $(1-i)^4$

12. $[\cos(\pi/5)+\sin(\pi/5)\,i]^{10}$

13. $\dfrac{1}{(\sqrt{3}+i)^6}$

14. $\dfrac{-2[\cos(\pi/3)-\sin(\pi/3)\,i]^3}{3+i}$

15. $\left(\dfrac{1+i}{2-2i}\right)^5$

16. $\dfrac{(2+i)^6}{(4-i)^3}$

17. What are the argument(s) of the complex number $z=0$?

18. Prove the following properties for moduli of complex numbers:

(a) $|z_1+z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}\,(z_1\overline{z_2}) + |z_2|^2$

(b) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

(c) $|z_1-z_2|^2 - |z_1+\overline{z_2}|^2 = -4(\mathrm{Re}\,z_1)(\mathrm{Re}\,z_2)$

19. Verify that for any complex number

$$|\mathrm{Re}\,z| + |\mathrm{Im}\,z| \le \sqrt{2}|z|.$$

§1.4 Exponential Form for Complex Numbers

As we mentioned at the end of the last section, the *exponential form* for complex numbers is so convenient, that we seldom use the polar form. You can almost see a connection with exponential functions in equations 1.23 and 1.25. If $r_1 = r_2 = 1$ in 1.23, then

$$(\cos\theta_1 + \sin\theta_1\, i)(\cos\theta_2 + \sin\theta_2\, i) = \cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\, i.$$

Compare this with $e^{\theta_1}e^{\theta_2} = e^{\theta_1+\theta_2}$. In each case we have something with θ_1 in it, multiplied by the same something with θ_2 in it, giving the same something with $\theta_1 + \theta_2$ in it. Similarly, compare equation 1.25 (with $r_1 = r_2 = 1$),

$$\frac{\cos\theta_1 + \sin\theta_1\, i}{\cos\theta_2 + \sin\theta_2\, i} = \cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\, i,$$

with $\dfrac{e^{\theta_1}}{e^{\theta_2}} = e^{\theta_1-\theta_2}$. We exploit this similarity by defining $e^{\theta i}$ for a real number θ as follows

$$e^{\theta i} = \cos\theta + \sin\theta\, i. \tag{1.29}$$

This is often called **Euler's identity**. From our perspective, it is not an identity; it is the definition of $e^{\theta i}$. We have worked with e^x for real x for many years. We have now defined what it means to take the exponential of a purely imaginary number. In Section 3.2, we will deal with the exponential function e^z for arbitrary z in which case, Euler's identity will be but a special case. However, because the exponential form for complex numbers has so many advantages over the polar representation, it would be a shame to delay its introduction. Using definition 1.29 of $e^{\theta i}$, we can write that

$$e^{\pi i/3} = \cos(\pi/3) + \sin(\pi/3)\, i = \frac{1}{2} + \frac{\sqrt{3}i}{2}, \qquad e^{\pi i/2} = \cos(\pi/2) + \sin(\pi/2)\, i = i.$$

If $z = r(\cos\theta + \sin\theta\, i)$ is the polar form of a complex number, then using 1.29, we may write

$$z = re^{\theta i}. \tag{1.30}$$

This is called the **exponential form** of a complex number. As for the polar form, it uses the modulus and argument of z, but replaces $\cos\theta + \sin\theta\, i$ with $e^{\theta i}$. In terms of this exponential representation, notice that equations 1.23 and 1.25 can be expressed in the forms

$$z_1 z_2 = (r_1 e^{\theta_1 i})(r_2 e^{\theta_2 i}) = r_1 r_2 e^{(\theta_1+\theta_2)i}, \tag{1.31a}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{\theta_1}}{r_2 e^{\theta_2}} = \frac{r_1}{r_2} e^{(\theta_1-\theta_2)i}. \tag{1.31b}$$

But this is exactly what we would expect if the complex exponential $e^{\theta i}$ were to obey the usual rules for real exponentials, namely that $e^{x_1}e^{x_2} = e^{x_1+x_2}$ and $e^{x_1}/e^{x_2} = e^{x_1-x_2}$. In other words, if we define $e^{\theta i}$ by 1.29, write complex numbers in form 1.30, and demand that $e^{\theta i}$ obey the usual multiplication and division rules, then equations 1.31 follow immediately, and we can forget about rules 1.23 and 1.25 for multiplication and division of complex numbers. DeMoivre's theorem is even more evident with exponential notation; it states that

$$z^n = (re^{\theta i})^n = r^n e^{n\theta i}. \tag{1.32}$$

Example 1.8 would now read as follows: Since $1 + \sqrt{3}i = 2e^{\pi i/3}$,

$$\begin{aligned}(1+\sqrt{3}i)^8 &= (2e^{\pi i/3})^8 = 2^8 e^{8\pi i/3} = 2^8 e^{2\pi i/3}\\ &= 256[\cos(2\pi/3)] + \sin(2\pi/3)\, i] = 256\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = 128(-1+\sqrt{3}i).\end{aligned}$$

We have replaced $e^{8\pi i/3}$ by $e^{2\pi i/3}$ and two simple arguments justify this. The easiest is to realize that $8\pi/3$ and $2\pi/3$ are arguments of a complex number and as such they are equivalent, $8\pi/3 = 2\pi + 2\pi/3$. Alternatively,

$$e^{8\pi i/3} = e^{(2\pi+2\pi/3)i} = e^{2\pi i}e^{2\pi i/3} = [\cos(2\pi) + \sin(2\pi)\, i]e^{2\pi i/3} = e^{2\pi i/3}.$$

What we are claiming here is stated more generally as

$$e^{(\theta+2k\pi)i} = e^{\theta i}, \qquad \text{whenever } k \text{ is an integer.} \tag{1.33}$$

We can also state that

$$e^{2k\pi i} = 1, \qquad \text{whenever } k \text{ is an integer.} \tag{1.34}$$

Example 1.9 Find the imaginary part of $\dfrac{(1+i)^{10}}{(\sqrt{3}-i)^5}$.

Solution Since

$$\frac{(1+i)^{10}}{(\sqrt{3}-i)^5} = \frac{(\sqrt{2}e^{\pi i/4})^{10}}{(2e^{-\pi i/6})^5} = \frac{2^5 e^{5\pi i/2}}{2^5 e^{-5\pi i/6}} = e^{(5\pi/2+5\pi/6)i} = e^{10\pi i/3} = e^{4\pi i/3},$$

the imaginary part is $\sin(4\pi/3) = -\sqrt{3}/2$.•

Example 1.10 Describe the position of the complex number $e^{\phi i}z$ in the complex plane relative to the position of z.

Solution If $z = re^{\theta i}$ is the exponential representation of z, then equation 1.31a gives

$$e^{\phi i}z = re^{(\phi+\theta)i}.$$

The modulus of z does not change, but its argument increases by ϕ. In other words, multiplying a complex number by $e^{\phi i}$ rotates it through angle ϕ about the origin.•

In some applications, it is advantageous to express $\cos n\theta$ and $\sin n\theta$, where $n \ge 2$ is an integer, as polynomials in $\sin\theta$ and $\cos\theta$. This can be done with trigonometric identities, but as the following example shows, DeMoivre's theorem provides an alternative.

Example 1.11 Use DeMoivre's theorem to verify that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\,\sin^2\theta.$$

Solution When the complex number in DeMoivre's theorem 1.32 has modulus $r = 1$,

$$z^n = (e^{\theta i})^n = e^{n\theta i}.$$

For $n = 3$, this becomes

$$(e^{\theta i})^3 = e^{3\theta i}.$$

If we replace the exponentials with their polar counterparts, we obtain

$$(\cos\theta + \sin\theta\, i)^3 = \cos 3\theta + \sin 3\theta\, i.$$

When the left side is expanded (with the binomial theorem), the result is

$$\begin{aligned}\cos 3\theta + \sin 3\theta\, i &= \cos^3\theta + 3\cos^2\theta(\sin\theta\, i) + 3\cos\theta(\sin\theta\, i)^2 + (\sin\theta\, i)^3 \\ &= (\cos^3\theta - 3\cos\theta\,\sin^2\theta) + (3\cos^2\theta\,\sin\theta - \sin^3\theta)i.\end{aligned}$$

Each side of this equation is a complex number, and two complex numbers are equal if and only if their real and imaginary parts are the same. When we equate real parts, we obtain

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\,\sin^2\theta.$$

Imaginary parts give the additional trigonometric identity

$$\sin 3\theta = 3\cos^2\theta\,\sin\theta - \sin^3\theta. \bullet$$

In other applications, it can prove advantageous to express $\cos^n\theta$ and $\sin^n\theta$, where $n \geq 2$ is an integer, as a polynomial in $\cos n\theta$ and $\sin n\theta$. Once again, this can be done with trigonometric identities, but complex exponentials can also be used. When we replace θ by $-\theta$ in Euler's identity 1.29, we obtain

$$e^{-\theta i} = \cos(-\theta) + \sin(-\theta)\, i = \cos\theta - \sin\theta\, i.$$

When this is added to Euler's identity, the result is

$$e^{\theta i} + e^{-\theta i} = 2\cos\theta.$$

In other words,

$$\cos\theta = \frac{e^{\theta i} + e^{-\theta i}}{2}. \qquad (1.35a)$$

Similarly,

$$\sin\theta = \frac{e^{\theta i} + e^{-\theta i}}{2}. \qquad (1.35b)$$

These can be used in examples such as the following.

Example 1.12 Use equation 1.35(a) to verify that

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta).$$

Solution If we raise both sides of the equation $\cos\theta = \dfrac{e^{\theta i} + e^{-\theta i}}{2}$ to power 4,

$$\cos^4\theta = \frac{1}{16}(e^{\theta i} + e^{-\theta i})^4 = \frac{1}{16}(e^{4\theta i} + 4e^{2\theta i} + 6 + 4e^{-2\theta i} + e^{-4\theta i})$$
$$= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6) = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta).\bullet$$

EXERCISES 1.4

In Exercises 1–6 express the complex number in exponential form.

1. $-1+i$ **2.** $-2i$ **3.** $\sqrt{3}-i$
4. $3+4i$ **5.** $-1-2i$ **6.** $-2[\cos(\pi/3) - \sin(\pi/3)\, i]$

In Exercises 7–19 express the complex number in simplified Cartesian form.

7. $3e^{\pi i/6}$ **8.** $e^{-\pi i}$ **9.** $(2e^{\pi i/4})^2$

10. $\dfrac{4e^{2\pi i/3}}{e^{\pi i/6}}$ **11.** $(1-i)^4$ **12.** $[\cos(\pi/5) + \sin(\pi/5)\, i]^{10}$

13. $\dfrac{1}{(\sqrt{3}+i)^6}$ **14.** $\dfrac{-2[\cos(\pi/3) - \sin(\pi/3)\, i]^3}{3+i}$ **15.** $\left(\dfrac{1+i}{2-2i}\right)^5$

16. $\dfrac{(2+i)^6}{(4-i)^3}$ **17.** $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ **18.** $\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 \left(\dfrac{1+i}{1-i}\right)^5$

19. $\dfrac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$

In Exercises 20–23 use DeMoivre's theorem 1.32 to verify the identity.

20. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

21. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

22. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

23. $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

In Exercises 24–26 use equations 1.35 to verify the identity.

24. $\cos^3\theta = \dfrac{1}{4}(3\cos\theta + \cos 3\theta)$

25. $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$

26. $\sin^4\theta = \dfrac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$

27. Show that if θ is an argument of a complex number z with modulus equal to one, then

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \qquad \sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right).$$

28. Verify that

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}.$$

In Exercises 29–30 use equations 1.35 to verify the identity when $n > 0$ is an integer.

29. $$\cos^n\theta = \begin{cases} \dfrac{n!}{2^n[(n/2)!]^2} + \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{(n-2)/2}\binom{n}{k}\cos(n-2k)\theta, & n \text{ even} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{(n-1)/2}\binom{n}{k}\cos(n-2k)\theta, & n \text{ odd.} \end{cases}$$

30. $$\sin^n\theta = \begin{cases} \dfrac{(-1)^n n!}{2^n[(n/2)!]^2} + \dfrac{(-1)^{n/2}}{2^{n-1}}\displaystyle\sum_{k=0}^{(n-2)/2}(-1)^k\binom{n}{k}\cos(n-2k)\theta, & n \text{ even} \\ \dfrac{(-1)^{(n+1)/2}}{2^{n-1}}\displaystyle\sum_{k=0}^{(n-1)/2}(-1)^{k+1}\binom{n}{k}\sin(n-2k)\theta, & n \text{ odd.} \end{cases}$$

§1.5 Applications of the Exponential Form for Complex Numbers

Roots of Complex Numbers

An equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \tag{1.36}$$

where $n \geq 1$ is an integer and $a_0, a_1, \ldots, a_n \neq 0$ are constants is called a **polynomial equation** of degree n. When the a_j are real, we speak of a real polynomial equation, and when the a_j are complex, we have a complex polynomial equation. For instance, the quadratic equation $3z^2 + 2z + 5 = 0$ is a real polynomial equation of degree 2; the equation $(4 - i)z^3 - 2z^2 + 3iz + 6 = 0$ is a complex polynomial equation of degree 3 (in short, a complex cubic equation). Early in your calculus studies, or even before, someone quoted to you a theorem called the Fundamental Theorem of Algebra. It was probably stated as follows: Every real polynomial equation of degree $n \geq 1$ has exactly n solutions (counting multiplicities). Recall the meaning of multiplicity. Each root $z = 1$ and $z = 2$ of the quadratic equation $z^2 - 3z + 2 = 0$ has multiplicity 1, but the root $z = 1$ of $z^2 - 2z + 1 = 0$ has multiplicity 2. The sixth degree equation $(z - 1)^2(z - 2)^3(z + 4) = 0$ has three distinct roots, but it has 6 roots counting multiplicities — $z = 1$ is a root of multiplicity 2, $z = 2$ has multiplicity 3, and $z = -4$ has multiplicity 1. The fundamental theorem indicates that when multiplicities are taken into account, the number of solutions is equal to the degree of the equation. The theorem is true for complex polynomial equations as well as real ones. You could not prove the real version when it was first given to you, and you cannot prove either version now, but you will be able to in Chapter 4. This presents no problem, since the theorem does not tell us how to find solutions anyway; it only tells us how many to expect.

Of particular interest are polynomial equations of the form

$$z^n = Z \tag{1.37}$$

where $n > 1$ is an integer, and Z is a given complex number. The n solutions are called the n n^{th} roots of Z. For example, the two solutions of $z^2 = 1 + i$ are the square roots of $1 + i$, and the three solutions of $z^3 = -1 + 2i$ are the cube roots of $-1 + 2i$. DeMoivre's theorem can be used to advantage here. To illustrate we begin with the example

$$z^3 = 8i.$$

The three solutions are called the cube roots of $8i$. To find them we express z and $8i$ in exponential form, $z = re^{\theta i}$ and $8i = 8e^{\pi i/2}$, and substitute into $z^3 = 8i$,

$$(re^{\theta i})^3 = 8e^{\pi i/2}.$$

Using DeMoivre's theorem on the left we obtain

$$r^3 e^{3\theta i} = 8e^{\pi i/2}.$$

Since conditions 1.21 for equality of complex numbers in polar form are also valid for complex numbers in exponential form, we can state that

$$r^3 = 8 \qquad \text{and} \qquad 3\theta = \frac{\pi}{2} + 2k\pi, \quad k \text{ an integer.}$$

Hence,

$$r = 2 \qquad \text{and} \qquad \theta = \frac{\pi}{6} + \frac{2k\pi}{3}.$$

The values $k = 0$, 1, and 2 lead to distinct complex numbers:

$$z_0 = 2e^{\pi i/6} = 2[\cos(\pi/6) + \sin(\pi/6)\,i] = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{3} + i,$$

$$z_1 = 2e^{(\pi/6+2\pi/3)i} = 2[\cos(5\pi/6) + \sin(5\pi/6)\,i] = 2\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right) = -\sqrt{3} + i,$$

$$z_2 = 2e^{(\pi/6+4\pi/3)i} = 2[\cos(3\pi/2) + \sin(3\pi/2)\,i] = -2i.$$

These are the cube roots of $8i$; the cube of each complex number is equal to $8i$. Other values of k simply repeat these cube roots. It is interesting to plot the cube roots in the complex plane (Figure 1.12). All three lie on a circle centred at the origin with radius 2 (the cube root of the modulus of $8i$). They are equally spaced around the circle with angle $2\pi/3$ between each pair. The argument of z_0 is one-third the principal argument of $8i$.

This process can be used on equation 1.37. If we set $z = re^{\theta i}$ and $Z = Re^{\Theta i}$, then

$$r^n e^{n\theta i} = Re^{\Theta i}.$$

Conditions 1.21 require

$$r^n = R \qquad \text{and} \qquad n\theta = \Theta + 2k\pi, \quad k \text{ an integer}.$$

Consequently,

$$r = R^{1/n} \qquad \text{and} \qquad \theta = \frac{\Theta + 2k\pi}{n}.$$

The n n^{th} roots of $Z = R(\cos\Theta + \sin\Theta\, i)$ are therefore

$$z_k = R^{1/n} e^{(\Theta+2k\pi)i/n} = R^{1/n}\left[\cos\left(\frac{\Theta+2k\pi}{n}\right) + \sin\left(\frac{\Theta+2k\pi}{n}\right) i\right], \quad (1.38)$$

where $k = 0, 1, \ldots, n-1$. Geometrically they are equally spaced around a circle of radius $R^{1/n}$ with angle $2\pi/n$ between successive pairs (Figure 1.13). The first (z_0) has argument Θ/n.

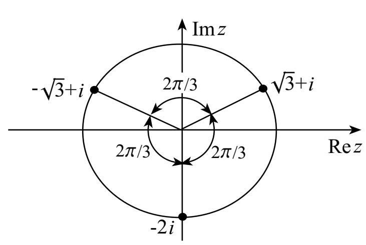


Figure 1.12

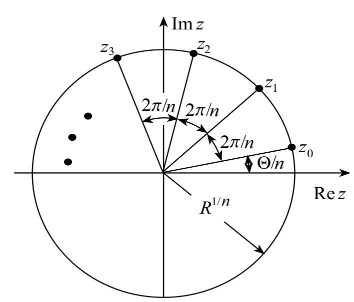


Figure 1.13

In Section 1.2 we found square roots of complex numbers. The above discussion provides a more efficient method. Compare the following discussion for finding the square roots of $2+i$ with that in Exercise 35 in Section 1.2. To find the square roots of $2+i$ we set $z = re^{\theta i}$ in $z^2 = 2+i$,

$$r^2 e^{2\theta i} = 2 + i = \sqrt{5}e^{\Theta i}$$

where $\Theta = \text{Tan}^{-1}(1/2)$. Conditions 1.21 require

$$r^2 = \sqrt{5} \qquad \text{and} \qquad 2\theta = \Theta + 2k\pi, \quad k \text{ an integer.}$$

Hence $r = 5^{1/4}$ and $\theta = \Theta/2 + k\pi$, and we obtain

$$z_0 = 5^{1/4}e^{\Theta i/2}, \qquad z_1 = 5^{1/4}e^{(\Theta/2+\pi)i} = -z_0.$$

The square roots are $\pm 5^{1/4}e^{\Theta i/2}$. We can use a calculator to approximate real and imaginary parts ($z \approx \pm(1.455 + 0.344i)$), or we can use trigonometry to find exact values. Since $\cos\Theta = 2/\sqrt{5}$ (Figure 1.14), it follows that

$$\sin(\Theta/2) = \sqrt{\frac{1-\cos\Theta}{2}} = \sqrt{\frac{\sqrt{5}-2}{2\sqrt{5}}} \quad \text{and} \quad \cos(\Theta/2) = \sqrt{\frac{1+\cos\Theta}{2}} = \sqrt{\frac{\sqrt{5}+2}{2\sqrt{5}}}.$$

Thus, the square roots are

$$\pm 5^{1/4}\left(\sqrt{\frac{\sqrt{5}+2}{2\sqrt{5}}} + \sqrt{\frac{\sqrt{5}-2}{2\sqrt{5}}}\,i\right) = \pm\left(\sqrt{\sqrt{5}/2+1} + \sqrt{\sqrt{5}/2-1}\,i\right).$$

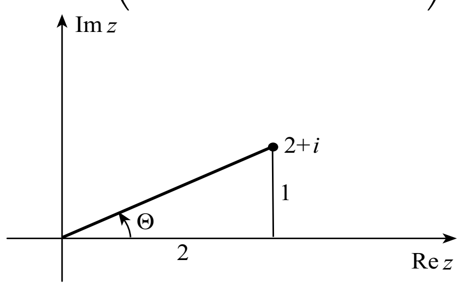


Figure 1.14

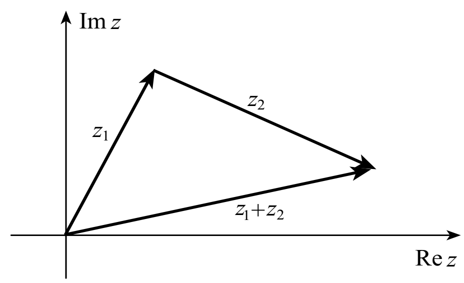


Figure 1.15

Triangle Inequality

It is a well-known fact that the length of any side of a triangle is less than the sum of the lengths of the other two sides (called the triangle inequality). If the sides of the triangle are represented by the complex numbers z_1, z_2, and $z_1 + z_2$ (Figure 1.15), then the triangle inequality can be expressed in terms of the moduli $|z_1|$, $|z_2|$, and $|z_1 + z_2|$,

$$|z_1 + z_2| < |z_1| + |z_2|. \tag{1.39}$$

When $z_2 = az_1$, where $a < 0$ is a real number (Figure 1.16), z_1, z_2, and $z_1 + z_2$ do not form a triangle, but inequality 1.39 is still valid. When $z_2 = az_1$, where $a > 0$ (Figure 1.17), there is once again no triangle, and in this case

$$|z_1 + z_2| = |z_1| + |z_2|.$$

Hence, we can say that for any two complex numbers z_1 and z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.40}$$

This continues to be called the triangle inequality in spite of the fact that when $z_2 = az_1$, there is no triangle. It is proved algebraically in Exercise 36.

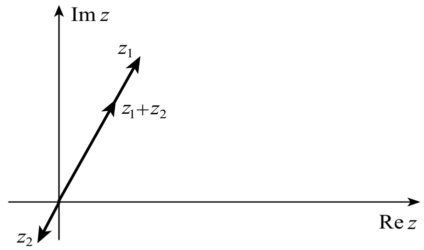


Figure 1.16

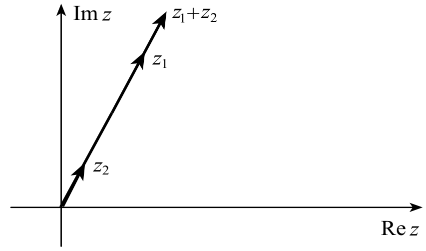


Figure 1.17

An alternative geometric statement to the triangle inequality (but perhaps not quite so obvious) is that the length of any side of a triangle is greater than the difference between the lengths of the other two sides. When the sides of the triangle are represented by z_1, z_2, and $z_1 + z_2$, this leads to the inequality

$$|z_1 + z_2| \geq \big| |z_1| - |z_2| \big|. \tag{1.41}$$

This result can also be proved algebraically or derived from 1.40 (see Exercises 37 and 38).

Ellipses and Circles

Moduli of complex numbers can be used to advantage in describing circles and ellipses in the complex plane (or the real plane). For example, consider the complex numbers that satisfy the inequality $|z-2+3i| \leq 4$. Vector subtraction suggests that $z - 2 + 3i = z - (2 - 3i)$ may be regarded as the vector from the complex number $2 - 3i$ to the complex number z (Figure 1.18). Because $|z - 2 + 3i|$ is the length of this vector, the inequality $|z - 2 + 3i| \leq 4$ describes all complex numbers z within a distance 4 from $2 - 3i$; that is, all points inside and on a circle centre $2 - 3i$ and radius 4. We can also see this algebraically. If we set $z = x + yi$ in $|z - 2 + 3i| \leq 4$, we obtain

$$4 \geq |(x + yi) - 2 + 3i| = |(x - 2) + (y + 3)i| = \sqrt{(x - 2)^2 + (y + 3)^2}.$$

When both sides are squared, $(x - 2)^2 + (y + 3)^2 \leq 16$. This describes a circle with centre $(2, -3)$ and radius 4, and its interior.

In general, for a fixed complex number z_0, the inequality $|z - z_0| \leq r$ describes all complex numbers inside and on a circle of radius r centred at z_0 (Figure 1.19).

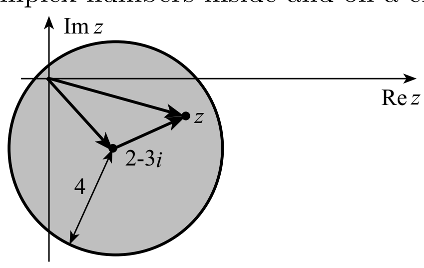


Figure 1.18

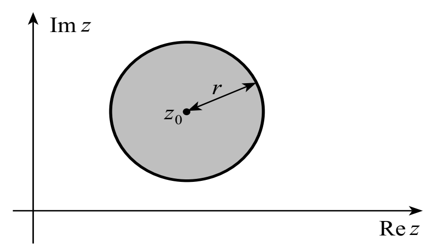


Figure 1.19

An ellipse is the path traced out by a point that moves so that the sum of its distances from two fixed points called foci is a constant value. If the foci are the complex numbers z_0 and z_1 (Figure 1.20), then the ellipse traced out by a point that moves so that the sum of its distances from z_0 and z_1 is equal to a is

$$|z - z_0| + |z - z_1| = a.$$

Naturally a must be larger than the distance between z_0 and z_1. For example, the equality $|z - 3i| + |z - 4| = 7$ describes the ellipse in Figure 1.21. The inequality $|z - 3i| + |z - 4| > 7$ describes all points outside the ellipse (Figure 1.22). The fact that the ellipse itself is not described by the inequality is indicated with a dashed curve.

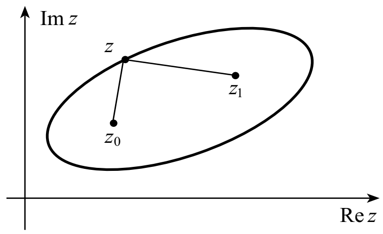


Figure 1.20

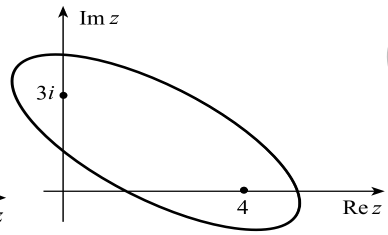


Figure 1.21

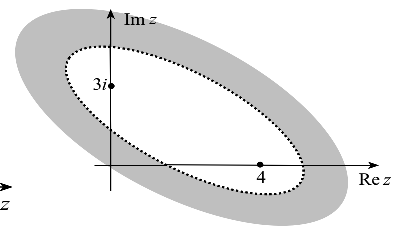


Figure 1.22

EXERCISES 1.5

In Exercises 1–3 use DeMoivre's theorem 1.32 to find the square roots of the complex number. Compare the results and the difficulty in arriving at these results relative to Exercises 33, 34, and 36 in Section 1.2.

1. $-7 - 24i$ **2.** $21 - 20i$ **3.** $-1 + i$

In Exercises 4–11 find all solutions of the equation.

4. $z^3 = 1$

5. $z^4 = i$

6. $z^2 = 2 + 2\sqrt{3}i$

7. $z^5 = -32$

8. $z^3 = 4 - 3i$

9. $z^6 = -1$

10. $z^4 + 3z^2 + 5 = 0$

11. $z^6 - 2z^3 + 4 = 0$

12. (a) The roots of the equation $z^n = 1$, where n is a positive integer, are called the **roots of unity**, often denoted by ω_k, $k = 0, \ldots, n-1$, where $\omega_0 = 1$. Find the other roots.

(b) Verify that

$$\omega_k \omega_j = \omega_r,$$

where r is the remainder when $j + k$ is divided by n.

(c) Prove that

$$\omega_0 + \omega_1 + \cdots + \omega_{n-1} = 0.$$

(d) Verify that for any $k = 1, \ldots, n-1$,

$$1 + \omega_k + \omega_k^2 + \cdots + \omega_k^{n-1} = 0.$$

(e) Prove that $k = 1, \ldots, n-1$,

$$1 + 2\omega_k + 3\omega_k^2 + 4\omega_k^3 + \cdots + n\omega_k^{n-1} = \frac{-n}{1 - \omega_k}.$$

13. Prove that when $|z_3| > |z_4|$,

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{|z_3| - |z_4|}.$$

Correct the inequality when $|z_3| < |z_4|$. What inequality encompasses both situations?

In Exercises 14–25 describe the set of points in the complex plane defined by the equation or the inequality.

14. $|z + i| = |z - 2i|$

15. $|z + 1| = 3|z + i|$

16. $|z - 2 + i| \leq 3$

17. $|z + 3i| > 2$

18. $1 \leq |z + 3| \leq 4$

19. $2 < |z - 2 - i| \leq 5$

20. $|z + 1| + |z - 1| \leq 4$

21. $|z - 2i| + |z + 3i| \geq 8$

22. $4 \leq |z - 1| + |z + 1| \leq 6$

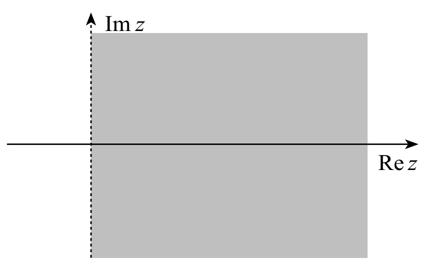
23. $z^2 + \overline{z}^2 = 4$

24. $z^2 + z + 2\overline{z} + 4 = 0$

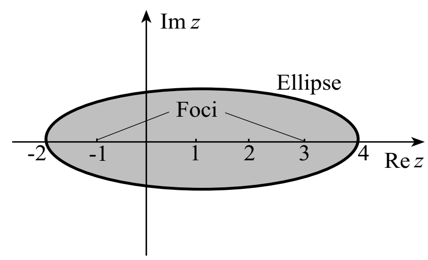
25. $|z| > 1 - \operatorname{Re} z$

In Exercises 26–33 describe the set of points algebraically.

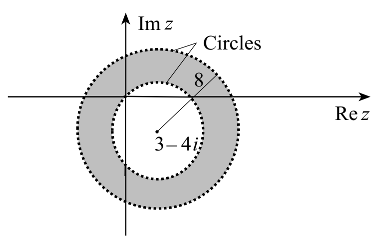
26.



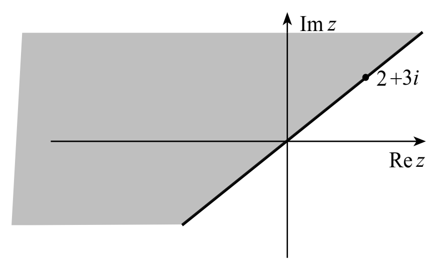
27.



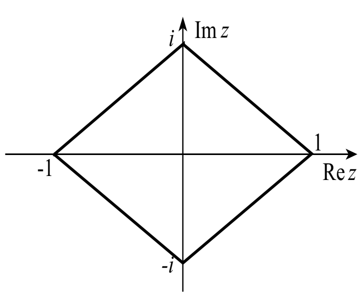
28.



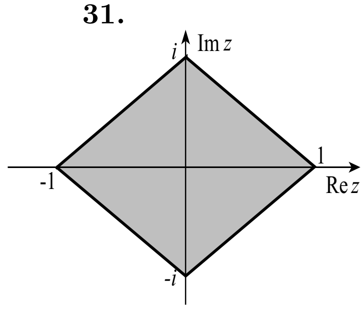
29.



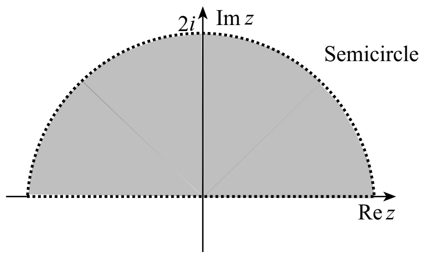
30.



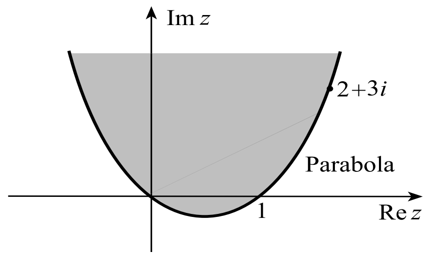
31.



32.



33.



34. Prove that the equation $|z - z_1| = |z - z_2|$ describes a line.

35. Use part (a) of Exercise 18 in Section 1.3 to show that when $\lambda > 0$ and $\lambda \neq 1$, the equation

$$|z - z_1| = \lambda|z - z_2|$$

describes a circle. Find its centre and radius.

36. Give an algebraic proof of triangle inequality 1.40 by setting $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

37. Give an algebraic proof of inequality 1.41 by setting $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

38. Use inequality 1.40 to prove inequality 1.41.

39. If Z is a given complex number and m and n are relatively prime, positive integers, find all solutions of the equation $z^n = Z^m$.

40. Show that for any real number p and any even, positive integer m,

$$e^{2mi\text{Cot}^{-1}p}\left(\frac{pi+1}{pi-1}\right)^m = 1.$$

41. Find all solutions of the equation $(z - 2 + 3i)^4 = z^4$.

42. Find all solutions of the equation $(z+3)^5 = z^5$. Hint: See Exercise 12.

43. In this exercise, we generalize the results of Exercises 41 and 42. Use Exercise 12 to show that for any nonzero complex number z_0, and any integer $n \geq 2$, solutions of the equation $z^n = (z + z_0)^n$ are

$$z = -\frac{z_0}{2}\left(1 + i\cot\frac{k\pi}{n}\right), \qquad k = 1, \ldots, n-1.$$

Confirm that this formula gives the solutions in Exercises 41 and 42.

44. Find all solutions of the equation

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Hint: Multiply the equation by $z - 1$.

45. Extend the result of the previous exercise to find all solutions of the equation

$$z^n + z^{n-1} + \cdots + z^2 + z + 1 = 0,$$

where n is a positive integer.

46. Prove that when $|z| < 1$, then for any positive integer n,

$$|1 + z + z^2 + z^3 + \cdots + z^n| < \frac{2}{1 - |z|}.$$