

THE FIELD GUIDE SERIES

for

MATHEMATICS

Combinatorics

Counting, Structure, Recurrence, and Connections

A UNIFIED VOLUME

LEARN - PRACTICE - INTEGRATE - RECOGNIZE

Randall J. Swift

THE FIELD GUIDE SERIES FOR MATHEMATICS

Combinatorics

Counting, Structure, Recurrence,
and Connections

A unified volume

Four coordinated modes in one book: Learn, Practice, Integrate, and Recognize. The volume preserves the visual maps, counting investigations, connection modules, and rapid-reference cards of the original set while giving them a common path through combinatorial structure.

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FIELD GUIDES FOR MATHEMATICS

The common architecture

LEARN

Field Guide

Essential ideas, maps, methods, examples, and connections.

PRACTICE

Field Workbook

Targeted calculation, investigation, explanation, and exploration.

INTEGRATE

Field Companion

Modules that connect theorems, recognition cues, and investigations.

RECOGNIZE

Field Cards

Rapid identification, formulas, decision cues, and visual reference.

Topics in the series

Trigonometry; Integration Techniques and Infinite Series; Vectors; Div, Grad, Curl and the Integral Theorems; Ordinary Differential Equations; Combinatorics; Probability Distributions; Mathematical Statistics; Real Analysis; Complex Analysis; and forthcoming volumes in limits, matrices, boundary-value problems, harmonic analysis, special functions, and stochastic processes.

Preface

Combinatorics is unusually unified. A single problem can be represented as a product, a binomial coefficient, a lattice path, a set partition, a generating function, a recurrence, or an orbit under symmetry. The art is to recognize which representation makes the structure visible.

This unified volume is designed as a compact companion to an undergraduate course in combinatorics. It follows the enumerative spine from modeling and elementary counting through permutations, stars and bars, bijections, Catalan and Stirling families, inclusion-exclusion, symmetry, generating functions, recurrences, probability, partitions, graphs, posets, q-analogues, designs, codes, and matroids.

The four modes are deliberately distinct but coordinated. Learn develops the main structures and visual maps. Practice turns recognition into calculation, explanation, and investigation. Integrate places the ideas into connected modules. Recognize provides a rapid reference for formulas, hypotheses, structural cues, and method selection.

The recommended path is Learn, then Practice, then Integrate, with Recognize available throughout as a quick reference. Across the volume, the recurring question is: what is one object, when are two descriptions the same, and which representation makes the count easiest to see?

How to use this volume

The four-mode study cycle

1

LEARN

Build the vocabulary and see the structural maps.

2

PRACTICE

Work investigations: Recognize -> Connect -> Compute.

3

INTEGRATE

Read modules that connect models, representations, and proofs.

4

RECOGNIZE

Use cards for rapid identification and method selection.

A practical routine

When a problem looks unfamiliar, begin with the model. Ask what the objects are, when two descriptions count as the same, whether order or repetition matters, and which representation matches the structure. Then use the field map or cards to select a method before calculating.

Recognize -> Connect -> Compute -> Interpret

The cards are a reference layer, not a substitute for hypotheses.

Notation and conventions

Throughout, $[n]$ denotes $\{1, \dots, n\}$; binomial coefficients count selections; $S(n,k)$ and $s(n,k)$ denote Stirling numbers of the second and unsigned first kinds; and $[x^n]F(x)$ extracts a coefficient. Unless stated otherwise, objects and boxes are labeled or unlabeled exactly as the problem specifies.

A composition remembers order, while an integer partition does not. A set partition divides labeled objects into unlabeled blocks. Generating functions are formal power series unless analytic convergence is explicitly relevant.

The volume separates modeling hypotheses from formulas. Before counting, identify the objects, labels, repetitions, order, restrictions, and equivalence relation. Then choose the representation: direct rule, complement, inclusion-exclusion, generating function, recurrence, symmetry, or probabilistic method.

A note on modeling hypotheses

A counting formula is only as good as its model. Check whether order matters, whether repetition is allowed, whether labels distinguish objects or boxes, and whether symmetry identifies multiple descriptions of the same object.

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The four parts are designed to be read sequentially or used independently.

A map of combinatorics

Combinatorics moves repeatedly between model, representation, structure, and proof. The same objects appear in the four modes with different questions: first learn the pattern, then work with it, then connect it, and finally recognize it quickly.

MODEL

objects, labels,
order, repetition



REPRESENT

paths, partitions,
generating functions



STRUCTURE

recurrences, graphs,
posets, designs



PROVE

bijections, symmetry,
probability, extremal ideas

Use Recognize whenever you need a fast return to the combinatorial map.

LEARN

Field Guide

The combinatorial vocabulary: models, counting rules, bijections, special families, algebraic encodings, graphs, and symmetry.

Counting begins with the model

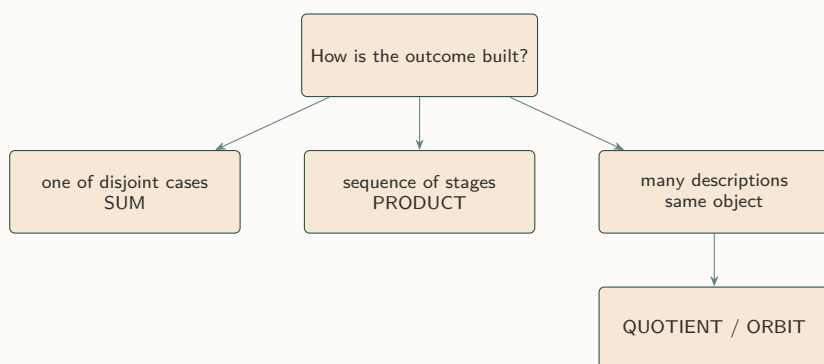
A formula is useful only after the outcome space is clear. A committee, a seating, a word, a function, and a partition may use the same underlying elements but impose different notions of equality.

The three elementary rules

Sum rule. Disjoint alternatives add: $|A \cup B| = |A| + |B|$ when $A \cap B = \emptyset$.

Product rule. Successive choices multiply: $n_1 n_2 \cdots n_k$.

Quotient rule. If every desired object has exactly m equivalent descriptions, divide the number of descriptions by m .



The first decision is structural: alternatives, stages, or equivalent descriptions.

Recognize: Words such as “either,” “case,” and “or” suggest a sum; “then,” “followed by,” and “for each” suggest a product. Rotations, reorderings, repeated labels, or indistinguishable components suggest a quotient.

Watch for: Do not divide by a symmetry unless every object has the same number of descriptions. Periodic necklaces, for example, can have nontrivial stabilizers; Burnside’s lemma is the safe replacement.

Complementary counting

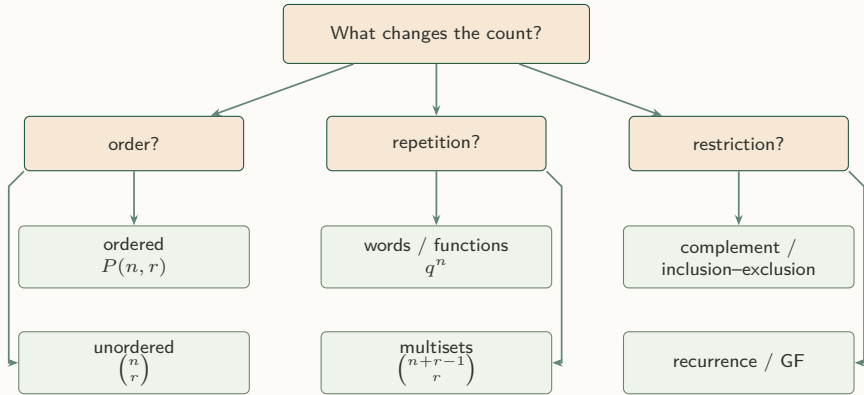
When a desired condition is “at least one,” the complement is often simpler:

$$|A| = |\Omega| - |A^c|.$$

For length- n strings over q symbols, the number containing a specified symbol at least once is $q^n - (q - 1)^n$.

Connections: Complementary counting is the one-event version of inclusion–exclusion; weighted product rules become generating-function multiplication.

A practical counting decision tree



Most elementary formulas are endpoints of modeling choices.

A constraint changes the model

How many four-digit positive integers have distinct digits and are divisible by 5? Split by last digit. Ending in 0 gives $9 \cdot 8 \cdot 7$; ending in 5 gives $8 \cdot 8 \cdot 7$. The answer is 952. The case split matters because a leading zero is forbidden.

Asymptotic scale

Exact counts can hide growth. Stirling’s approximation

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

gives, for example,

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \quad C_n \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}.$$

Recognize: Use asymptotics when the main question is scale, threshold, or comparison rather than an exact integer.

Ordered and unordered selections

From n distinct objects:

$$P(n, r) = n^{\underline{r}} = \frac{n!}{(n-r)!}, \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

The quotient by $r!$ forgets the order of the selected r objects.



ordered selection
 $A \rightarrow B \rightarrow D$

unordered selection
 $\{A, B, D\}$

Permutation records a sequence; combination records membership only.

Functions as counting objects

If $|A| = m$ and $|B| = n$:

$$\#\{f : A \rightarrow B\} = n^m, \quad \#\{\text{injections}\} = n^{\underline{m}},$$

$$\#\{\text{surjections}\} = n! \left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \sum_{j=0}^n (-1)^j \binom{n}{j} (n-j)^m.$$

This is a useful bridge: one surjection can be counted either by its missed values (inclusion–exclusion) or by its nonempty fibers (Stirling numbers).

Circular arrangements

For n distinct objects around a circle, rotations are identified, so there are $(n-1)!$ circular orders. If reflection is also identified and no nontrivial symmetry remains, divide by 2 to obtain $(n-1)!/2$ for $n \geq 3$.

Recognize: Before counting a circular object, decide which motions are allowed: rotation only, or rotation and reflection? Repeated colors require orbit counting rather than naive division.

Repeated objects and multinomial orderings

If $n = n_1 + \cdots + n_k$, the number of words with n_i copies of symbol i is

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! \cdots n_k!}.$$

The same coefficient counts assignments of n distinct objects to k labeled groups of prescribed sizes.

Repeated letters

BALLOON has seven letters with two L's and two O's. Hence

$$\frac{7!}{2!2!} = 1260$$

distinct arrangements.

Pairings

The number of ways to partition $2n$ labeled objects into n unordered pairs is

$$\frac{(2n)!}{2^n n!} = (2n-1)!!.$$

A second derivation chooses the partner of object 1 and recurses:

$$M_{2n} = (2n-1)M_{2n-2}, \quad M_0 = 1.$$

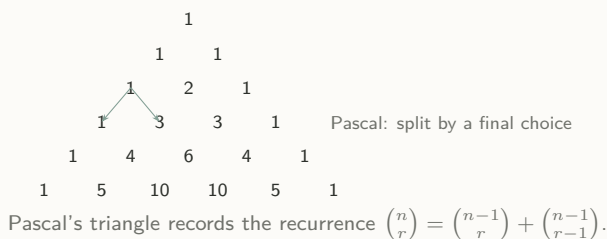
Connections: Pairings lead in two directions: forbidding crossings produces Catalan numbers; allowing singletons produces involutions with EGF $\exp(x + x^2/2)$.

The binomial coefficient is simultaneously a subset count, a path count, a polynomial coefficient, and an entry of Pascal's triangle. That multiplicity of interpretations is why binomial identities are so portable.

Binomial theorem

$$(x + y)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r.$$

The coefficient $\binom{n}{r}$ records which r of the n factors contribute y .



Identities to recognize

$$\binom{n}{r} = \binom{n}{n-r}, \quad \sum_r \binom{n}{r} = 2^n,$$

$$\sum_r r \binom{n}{r} = n2^{n-1}, \quad \sum_k \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.$$

The last identity is Vandermonde's convolution: choose an r -subset from two disjoint groups.

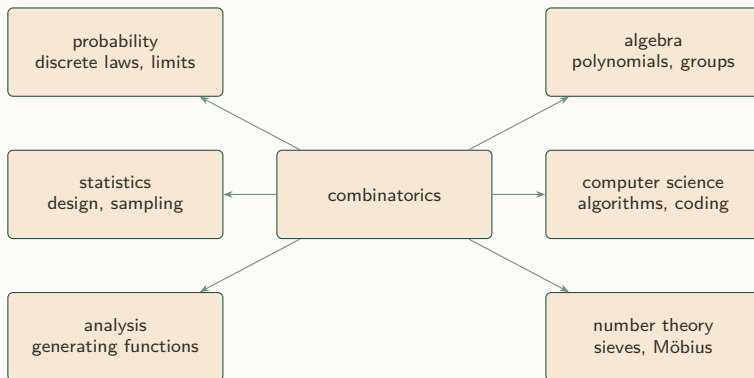
Nested selection

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}.$$

Both sides count pairs $K \subseteq R \subseteq [n]$ with $|K| = k$ and $|R| = r$.

Where next?

How this volume connects across mathematics



Combinatorics is both a subject and a transferable way of representing structure.

The strongest continuation depends on which representation has become most useful. Probability develops random occupancy, the probabilistic method, random graphs, and limiting laws. Algebra deepens group actions, q -analogues, symmetric functions, matroids, and algebraic combinatorics. Analysis leads to analytic combinatorics and asymptotic coefficient extraction. Computer science adds algorithms, complexity, coding, networks, and optimization. Number theory develops sieve methods, partitions, and additive structure. Graph theory, extremal combinatorics, posets, designs, and finite geometry each provide major structural branches in their own right.

The conceptual handoff

counting rules $\longrightarrow$ combinatorial proof $\longrightarrow$ special structures
 $\longrightarrow$ symmetry / recurrences / generating functions
 $\longrightarrow$ probability, graph theory, number theory, algebra, and algorithms.

PRACTICE

Field Workbook

Targeted investigations that ask readers to identify the object, representation, and structure before calculating.

Modeling and the Elementary Counting Rules

Field Investigation 1.1 — Model, calculate, interpret

A password consists of six symbols chosen from ten digits and four letters. Exactly two positions must contain letters, repetitions are allowed, and the first position cannot be 0.

1. State precisely what constitutes one outcome.
2. Count by splitting according to whether the first position is a letter or digit.
3. Recount by choosing the letter positions first and reconcile the two expressions.
4. Identify where the sum rule and product rule enter.

Field Cards: A01, V01.

Field Check

The same problem can usually be counted in more than one order. A correct solution makes the model and the equivalence of descriptions explicit.

More Practice

Field Investigation 1.2 — Change the representation

For each situation, decide first whether the structural cue is a sum, product, quotient, complement, or a combination of them.

1. Count length-8 ternary words containing at least one 2.
2. Count ways to choose a chair, secretary, and three-person committee from ten people when the chair and secretary cannot be on the committee.
3. A necklace has eight distinct beads and rotations are identified. Explain why naive division by 8 works here, and why the same argument can fail with repeated colors.

Further Exploration

Invent a counting problem for which changing the order of choices turns an awkward case split into a simple product.

Selections, Functions, and Circular Arrangements

Field Investigation 2.1 — Model, calculate, interpret

Let A have 7 elements and B have 4 elements.

1. Count all functions $A \rightarrow B$.
2. Count the surjections $A \rightarrow B$ two ways: by inclusion–exclusion and by nonempty fibers using Stirling numbers.
3. Count injections $B \rightarrow A$.
4. Explain how the formulas distinguish ordered choices, unordered selections, and functions.

Field Cards: A02–A04.

Field Check

Surjections are a useful bridge: missed codomain values lead to inclusion–exclusion, while nonempty fibers lead to set partitions.

More Practice

Field Investigation 2.2 — Change the representation

Work with the multiset of letters in BALLOON.

1. Count distinct linear arrangements.
2. Count arrangements beginning with a vowel.
3. If the seven positions are arranged on a circle and rotations are identified, explain why simply dividing the linear answer by 7 requires care.
4. Compare with the circular count when all seven symbols are distinct.

Further Exploration

Construct an example with repeated colors where some circular configurations have nontrivial stabilizers.

INTEGRATE

Field Companion

Connected modules that place modeling, representation, proof, computation, and interpretation together.

Model First: Sum, Product, Quotient, and Selection

LEARN — Essential structure

Combinatorial counting starts with an outcome model. Sum rules encode disjoint alternatives, product rules encode successive stages, and quotient rules forget equivalent descriptions only when every desired object has the same number of descriptions. Permutations, combinations, and functions are consequences of these modeling choices, not substitutes for them.

RECOGNIZE — What should trigger the method?

- either/case split → sum rule;
- successive independent choices → product rule;
- ordered versus unordered selection → permutation versus combination;
- circular or symmetric descriptions → quotient only after stabilizers are understood;

Field Cards: A01–A04, V01.

PRACTICE — Field investigation

Field Investigation 1.1 — Integrate the method

A seven-character code uses digits and letters, contains exactly three letters, and begins with a nonzero digit. Count it two ways, making the first modeling decision explicit in each approach. Then modify the problem so repeated symbols are forbidden and explain which factor changes.

Field Check

The calculation should be readable as a sequence of structural decisions: outcome type, cases, positions, then symbols.

Change Representation: Paths, Occupancy, and Proof

LEARN — Essential structure

Many identities become simple after a representation change. Subsets are binary words and monotone lattice paths; weak compositions are stars-and-bars strings; a double count marks an incidence and counts it in two orders; pigeonhole arguments compare the size of an avoiding configuration with the next forced object.

RECOGNIZE — What should trigger the method?

- integer solutions $\rightarrow$ stars and bars or coefficient extraction;
- binomial identity $\rightarrow$ subset/path/double-count model;
- forced collision $\rightarrow$ pigeonhole;
- balanced row/column repetition $\rightarrow$ Latin-square or design viewpoint;

Field Cards: A05–A10.

PRACTICE — Field investigation

Field Investigation 2.1 — Integrate the method

Count monotone paths from $(0, 0)$ to $(9, 6)$ that pass through $(4, 2)$. Recast the answer as a subset count, then derive a Vandermonde-style identity by summing over one intermediate coordinate.

Field Check

The same coefficient should emerge from geometry and from choosing positions. The point is the equivalence of encodings.

RECOGNIZE

Field Cards

Rapid identification cards for hypotheses, formulas, structural cues, and method selection.

Counting Begins with the Model

Recognize: Decide what one outcome is and when two descriptions are the same before choosing a formula.

Core relation

$|A \sqcup B| = |A| + |B|, \qquad |A \times B| = |A||B|.$

- Sum rule

Disjoint alternatives add.
- Product rule

Successive independent choices multiply.



Counting Begins with the Model

Quotient rule. Divide only when every desired object has the same number of descriptions.

Complement. Count all objects minus the forbidden class when that is simpler.

Warning. Unequal stabilizers make naive division by symmetry unsafe.

Use it for

choosing the first structural rule in a counting problem.

Connections

Sum/product rules become generating-function addition/multiplication; quotient ideas lead to orbit counting.

Permutations and Combinations

Recognize: Ask whether order matters and whether repetition is allowed.

Core relation

$$P(n, r) = \frac{n!}{(n-r)!}, \qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

- Ordered**
- Use permutations when position matters.
- Unordered**
- Use combinations when only membership matters.



Permutations and Combinations

Repetition. Words of length r over n symbols: n^r .

Symmetry. $\binom{n}{r} = \binom{n}{n-r}$.

Model cue. A selection formula is the endpoint of modeling choices, not the starting point.

Use it for

sampling, committees, strings, and elementary coefficient counts.

Connections

Binomial coefficients also count lattice paths and polynomial coefficients.

