

**Theory and Applications of
Ordinary Differential Equations**

**With an Introduction to
Partial Differential Equations**

Donald W. Trim

The University of Manitoba

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CHAPTER 1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

In physical systems, and even in systems that are not physical, a change in one quantity often precipitates changes in other quantities. Relationships among the rates at which these changes take place lead to what are called *differential equations*, and the number of areas of applied mathematics that give rise to differential equations is truly amazing. The fact that they serve as mathematical models not only in the physical sciences of engineering, physics, and chemistry, but also in less traditional areas such as economics, medicine, and music, indicates why they are worthy of our study. In this chapter we give various classifications of differential equations along with some simple examples to illustrate the direction of investigations in the remainder of the text.

1.1 Classification of Differential Equations

You have likely solved hundreds, even thousands, of equations in your mathematical studies. You have solved algebraic equations like $x^3 + 5x^2 - x - 5 = 0$ for values $x = -5, -1, 1$. You have also solved equations like $x^3y + x^2 = 5$ and $e^y - e^{-y} = 2x$ that define y implicitly as functions of x for explicit definitions of the functions, namely, $y = (5 - x^2)/x^3$ and $y = \ln(x + \sqrt{x^2 + 1})$. Differential equations must also be solved for functions, but unlike the above equations, differential equations contain one or more derivatives of the unknown functions. This makes them much more formidable, but at the same time, far more interesting. A **differential equation** then is an equation containing derivatives of unknown functions that must be solved for the functions. For example, each of the following equations is a differential equation in y as a function of x , except in equation 1.1e, where x is a function of time t :

$$\frac{dy}{dx} = ky \left(1 - \frac{y}{C}\right), \quad (k \text{ and } C \text{ constants}), \quad (1.1a)$$

$$\frac{d^2y}{dx^2} = k \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad (k \text{ a constant}), \quad (1.1b)$$

$$xy'' + y' + xy = 0, \quad (1.1c)$$

$$\frac{d^4y}{dx^4} - k^4y = 1, \quad (k \text{ a constant}), \quad (1.1d)$$

$$m\ddot{x} + \beta\dot{x} + kx = \sin t, \quad (m, \beta, \text{ and } k \text{ constants}), \quad (1.1e)$$

$$y^{(10)}(x) + 4xy^{(5)}(x) - 2x\frac{dy}{dx} = \sin x. \quad (1.1f)$$

Equation 1.1a is used to determine sizes of populations which follow the *logistic model* (Section 3.3); equation 1.1b describes shapes of hanging cables (Section 2.8); equation 1.1c, called Bessel's differential equation of order zero, is found in heat flow and vibration problems (Section 9.9); and equation 1.1d is used to determine deflections of beams (Section 5.5). Engineers often indicate derivatives with respect to time with dots above a function as in equation 1.1e. One dot in $\dot{x}$ represents dx/dt , and two dots in $\ddot{x}$ means d^2x/dt^2 . This equation arises in Section 5.1 when we study vibrating mass-spring systems. Equation 1.1f has no applications that we are aware of. We have listed it to introduce the notation $y^{(10)}(x)$ which is sometimes shortened

to $y^{(10)}$; it represents the tenth derivative of the function $y(x)$ with respect to x . A superscript enclosed in parentheses on a function always indicates a derivative of the function, not a power of the function. Likewise then, $y^{(5)}(x)$ is the fifth derivative of $y(x)$ with respect to x . The notation y' and y'' in equation 1.1c is the third way to represent derivatives; they denote the first and second derivatives of y with respect to x .

We could be more explicit in showing that a differential equation is an equation that must be solved for functions by writing equation 1.1c in the form

$$xy''(x) + y'(x) + xy(x) = 0.$$

Because this makes the differential equation more cumbersome to write, and to look at, we will refrain from using this representation, unless there is a special reason for doing so. The variable being differentiated is always dependent, depending on the other variable in the equation.

We customarily use x to denote the independent variable and y to denote the dependent variable when discussing differential equations, as in equations 1.1a,b,c,d,f. In applications, we use letters that reflect quantities that they represent. For instance, in differential equation 1.1e, x represents displacement of the mass m from its equilibrium position, and t denotes time. If temperature at points in a sphere depends only on distance from the centre of the sphere, we would use r to represent this distance, and T to represent temperature.

Differential equations can be classified in various ways, as *ordinary* or *partial*, as *linear* or *nonlinear*, and as to *order*. When a differential equation is to be solved for a function of only one variable, as in equations 1.1, it is called an **ordinary** differential equation. When y is a function of more than one variable, say x and t , and the differential equation contains partial derivatives of y with respect to x and/or t , we have a **partial** differential equation. Examples of partial differential equations are

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad (1.2a)$$

called the *one-dimensional diffusion equation*, and

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad (1.2b)$$

the *one-dimensional wave equation*. Another important partial differential equation is *Poisson's equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = f(x, y, z), \quad (1.2c)$$

for $V(x, y, z)$. A special case of Poisson's equation occurs when $f(x, y, z) = 0$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (1.2d)$$

It is called Laplace's equation. Chapters 1–10 deal with ordinary differential equations; Chapters 11–14 tackle partial differential equations. With this in mind, we

shall refer to *ordinary differential equations* in Chapters 1–10 simply as *differential equations*.

Differential equations are also classified according to the highest order derivative contained therein.

Definition 1.1 The **order** of a differential equation is the order of the highest derivative in the equation.

The first of differential equations 1.1 is first-order, the second, third, and fifth are second-order, the fourth is fourth-order, and the last is tenth-order. Each partial differential equation 1.2a–d is second-order.

The third way to distinguish between differential equations is most important; it classifies them as *linear* or *nonlinear*. You have already encountered the concept of “linearity” in various settings. First, you will recall that a linear equation in x and y is one of the form $ax + by = c$, where a , b , and c are constants; it describes a line in the xy -plane. We say that the left side of the equation is a linear combination of x and y . The equation $Ax + By + Cz = D$, where coefficients are again constants, is linear in x , y , and z ; it describes a plane in xyz -space. The left side of the equation is a linear combination of x , y , and z . We can write a vector $\mathbf{v}$ with Cartesian components (v_x, v_y, v_z) in the form $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit basis vectors in the x -, y -, and z -directions, respectively. This expresses $\mathbf{v}$ as a linear combination of the basis vectors. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are vectors, the vector $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$, where the c_i are constants is called a linear combination of vectors $\mathbf{v}_i$. With these ideas recalled, we now define what we mean by a linear differential equation.

Definition 1.2 An n^{th} -order differential equation in $y(x)$ is said to be **linear** if it can be expressed in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x). \quad (1.3)$$

The left side of the equation is a linear combination of $y(x)$ and its n derivatives. This is one reason for calling the equation linear; a more important reason will be discussed in Section 4.1. Coefficients $a_i(x)$ can be numbers, but they can also be functions of x (but not functions of y). Notice in particular that none of the derivatives of $y(x)$ are multiplied together, nor are they squared or cubed or taken to any other power, nor do they appear as the argument of any transcendental function. There is a function of x multiplying y , **plus** a function of x multiplying the first derivative of y , **plus** a function of x multiplying the second derivative of y , and so on, to the n^{th} derivative of y . In order that the equation be n^{th} -order, we assume that $a_n(x)$ is not identically equal to zero. It could be equal to zero at isolated values of x , but not equal to zero for all x . For example, linear first- and second-order equations are ones that can be expressed in the forms

$$a_1(x)\frac{dy}{dx} + a_0(x)y = F(x), \quad a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x).$$

The following equations are nonlinear.

$$x\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + \sin y = e^{2x}, \quad \frac{d^3 y}{dx^3} \frac{dy}{dx} - 2y = x^2.$$

The first equation is nonlinear because of the $\sin y$ -term; the second is nonlinear due to the product of d^3y/dx^3 and dy/dx . Differential equations 1.1c,d,e,f are linear; equations 1.1a,b are nonlinear.

Example 1.1 Determine whether the following differential equations are linear:

$$(a) \sin x \frac{d^2y}{dx^2} + 3e^{2x}y^2 = 6x \quad (b) y'' + e^xy' + xy = \sin 2x \quad (c) (1 + y')^{1/3} + x^2 = 0$$

Solution (a) The $3e^{2x}y^2$ in this equation makes the equation nonlinear, not because of e^{2x} , but because of y^2 .

(b) This equation is linear; it is of form 1.3. Coefficient e^x , and $\sin 2x$, are not linear functions of x , but that is irrelevant. The left side of the equation is a linear combination of y , y' and y'' .

(c) Because we can rewrite this equation in the form $\frac{dy}{dx} = -1 - x^6$, it is linear. •

Solutions of Differential Equations

Differential equations must be solved for functions. A function is a solution of a differential equation if it satisfies the following requirement.

Definition 1.3 A function is a **solution** of a differential equation on an open interval[†] I if substitution of the function into the differential equation reduces the differential equation to an identity on the interval.^{††}

What we mean by saying that the differential equation is “reduced to an identity” is that when the proposed solution is substituted into the differential equation, left and right sides of the equation become equal at every point in the interval I . For example, to show that the function $y(x) = e^{5x} - 2e^{-x} + x^2 + 2x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = -5x^2 - 18x - 6$$

on the interval $-\infty < x < \infty$, we substitute it into the left side of the differential equation,

$$\begin{aligned} \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y &= (25e^{5x} - 2e^{-x} + 2) - 4(5e^{5x} + 2e^{-x} + 2x + 2) \\ &\quad - 5(e^{5x} - 2e^{-x} + x^2 + 2x) \\ &= -5x^2 - 18x - 6. \end{aligned}$$

Since this is the right side of the differential equation, we have verified that substitution of the function into the differential equation reduces it to an identity on the interval $-\infty < x < \infty$.

[†] By considering differential equations on open intervals we avoid one-sided derivatives at the ends of the interval.

^{††} This is a text on **real** differential equations, and therefore we seek real-valued solutions. We may sometimes find it useful to introduce complex numbers, but ultimately we want real solutions.

Example 1.2 Show that $y(x) = C_1 \cos 3x + C_2 \sin 3x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} = -9y$$

on the interval $-\infty < x < \infty$ for any constants C_1 and C_2 .

Solution If we substitute the function into the left side of the differential equation, we get

$$\frac{d^2y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x.$$

Substitution into the right side of the equation gives

$$-9y = -9(C_1 \cos 3x + C_2 \sin 3x).$$

Since these expressions are equal for all x and any C_1 and C_2 , the given function is indeed a solution of the differential equation on the interval $-\infty < x < \infty$. Because $y(x) = C_1 \cos 3x + C_2 \sin 3x$ contains two arbitrary constants, it is not just a solution, but a double-infinity of solutions. We call it a **2-parameter family of solutions**.•

Example 1.3 What is the largest interval on which $y(x) = x^{-1/2} \sin x$ is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0?$$

Solution Substitution of the function into the left side of the differential equation gives

$$\begin{aligned} x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y &= x^2 \left(-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{3/2}} + \frac{3 \sin x}{4x^{5/2}} \right) \\ &\quad + x \left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}} \right) + \left(x^2 - \frac{1}{4}\right) \frac{\sin x}{\sqrt{x}} \\ &= \sin x \left(-x^{3/2} + \frac{3}{4\sqrt{x}} - \frac{1}{2\sqrt{x}} + x^{3/2} - \frac{1}{4\sqrt{x}} \right) \\ &\quad + \cos x (-\sqrt{x} + \sqrt{x}) \\ &= 0. \end{aligned}$$

Consequently the function $y(x) = x^{-1/2} \sin x$ is a solution of the differential equation and is so for all values of x for which differentiations make sense; namely, $x > 0$.•

Finding Versus Checking Solutions to Differential Equations

There is a vast difference between *checking* whether a function is a solution of a differential equation, and *finding* a solution of a differential equation. If we have a function that we believe to be a solution of a differential equation, it is relatively simple to check whether this is indeed the case. As in Examples 1.2 and 1.3, we simply substitute the function into the differential equation to see if it reduces the equation to an identity. This means that you should **never** give an incorrect solution to a differential equation. You can **always** check whether it is correct.

Finding solutions to a differential equation can sometimes be a formidable task, even an impossible one. Much of this text is devoted to finding solutions of certain types of differential equations, but the reader should realize that many equations, especially those that arise in engineering, physics, economics, etc, may not fall into these categories. Particularly troublesome are nonlinear equations; to find exact solutions, it may be necessary to resort to methods beyond the scope of this book. Alternatively, approximate solutions can be derived with the methods in Chapters 9 and 10.

Intervals of Validity for Solutions to Differential Equations

When differential equations arise in applications, intervals on which solutions are desired are usually known. For instance, differential equation 1.1b describes the shape of a hanging cable, and therefore it would be solved on the open interval $0 < x < L$, given that the ends of the cable are at $x = 0$ and $x = L$. Equation 1.1e describes the displacement of a mass on the end of a spring, and it would be solved for $t > 0$, given that the motion of the mass commences at time $t = 0$. When differential equations do not arise from applications, we often find solutions, and subsequently determine their domains of validity. For instance, it is easy to see (or check) that exponential functions $y(x) = Ce^{2x}$ are solutions of the differential equation $dy/dx = 2y$ for any value of the constant C , and they are all solutions on the interval $-\infty < x < \infty$. It sometimes happens, however, that different solutions of a differential equation are defined on different intervals. For example, elementary integration suggests that functions that satisfy the differential equation

$$\frac{dy}{dx} = \frac{1}{(x+2)^2}, \quad (1.4)$$

are of the form

$$y(x) = \frac{-1}{x+2} + C, \quad (1.5a)$$

where C is an arbitrary constant. But where is this function a solution of differential equation 1.4? Clearly the differential equation need not be considered at $x = -2$. The solution is valid on the interval $x < -2$, or the interval $x > -2$, or any subinterval of these intervals, but not on any interval that contains $x = -2$. A better description of solutions is

$$y(x) = \begin{cases} -\frac{1}{x+2} + C_1, & x < -2 \\ -\frac{1}{x+2} + C_2, & x > -2 \end{cases} \quad (1.5b)$$

where C_1 and C_2 are arbitrary (unrelated) constants. Expression 1.5b contains solutions of differential equation 1.4 on the interval $x < -2$, and solutions on the interval $x > -2$, but they are completely independent solutions. Solutions have the same form on these intervals, but they are unrelated. We often use compact notation 1.5a for solutions of differential equations like 1.4, but realize that further analysis of the differential equation and its solutions might necessitate a division of 1.5a into its component parts 1.5b.

Example 1.4 Find solutions of the differential equation

$$(x^2 - 1) \frac{dy}{dx} = x.$$

Solution When we write the differential equation in the form

$$\frac{dy}{dx} = \frac{x}{x^2 - 1},$$

it can immediately be integrated to give

$$y(x) = \frac{1}{2} \ln |x^2 - 1| + C.$$

The solution should clearly not be considered at $x = \pm 1$. It is valid on the intervals $-\infty < x < -1$, $-1 < x < 1$, and $1 < x < \infty$. As a result, the best way to write the solution is

$$y(x) = \frac{1}{2} \begin{cases} \ln(x^2 - 1) + C_1, & -\infty < x < -1 \\ \ln(1 - x^2) + C_2, & -1 < x < 1 \\ \ln(x^2 - 1) + C_3, & 1 < x < \infty. \end{cases}$$

We have an infinity of solutions of the differential equation on the interval $-\infty < x < -1$, an infinity of solutions on the interval $-1 < x < 1$, and an infinity of solutions on the interval $1 < x < \infty$.•

Implicitly Defined Solutions

In the differential equations that we have considered, x has been the independent variable and y the dependent variable. In every case, solutions have been defined explicitly; that is, solutions have been written in the form $y = y(x)$. Because functions can be defined implicitly as well as explicitly, it should not be surprising that solutions of a differential equation might be defined implicitly rather than explicitly. For example, we have plotted the curve $5y^3 + x^3 + x - 10y = 1$ in Figure 1.1. It defines three functions, one being that part of the curve above the point P , a second being the part of the curve between P and Q , and the third being the part of the curve below Q . We claim that each of these functions satisfies the nonlinear differential equation

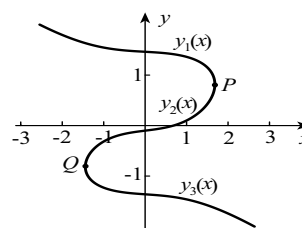


Figure 1.1

$$\frac{dy}{dx} = \frac{3x^2 + 1}{10 - 15y^2}.$$

To prove this, we differentiate the equation of the curve implicitly with respect to x ,

$$15y^2 \frac{dy}{dx} + 3x^2 + 1 - 10 \frac{dy}{dx} = 0.$$

When this is solved for dy/dx , we obtain

$$\frac{dy}{dx} = \frac{3x^2 + 1}{10 - 15y^2},$$

the given differential equation. All three of the functions defined by the equation $5y^3 + x^3 + x - 10y = 1$ and shown in Figure 1.1 satisfy the differential equation, but they do so on different intervals. The uppermost function $y_1(x)$ satisfies the differential equation on the interval $-\infty < x < x_P$, where x_P is the x -coordinate of P ; the middle function $y_2(x)$ satisfies the differential equation on the interval $x_Q < x < x_P$, where x_Q is the x -coordinate of Q ; and the lowest function $y_3(x)$ is a solution on the interval $x_Q < x < \infty$. We cannot find these functions explicitly, as it is impossible to solve $5y^3 + x^3 + x - 10y = 1$ for y in terms of x , but in Figure 1.1, we have their graphs. Each graph is called a **solution curve** of the differential equation.

The differential equation identifies the points P and Q . At these points, the slope of the curve is undefined, and this occurs when $10 - 15y^2 = 0$. Thus, y -coordinates of P and Q are $\pm\sqrt{2/3}$. Corresponding x -coordinates can be obtained by substituting these values of y into $5y^3 + x^3 + x - 10y = 1$,

$$\pm 5(2/3)^{3/2} + x^3 + x \mp 10(2/3)^{1/2} = 1.$$

These equations can be solved numerically for $x_P \approx 1.68$ and $x_Q \approx -1.44$.

We formalize these ideas in the following definition.

Definition 1.4 An equation $f(x, y) = 0$ is said to **implicitly** define a solution of a differential equation on an open interval I if there exists a function, defined implicitly by the equation, that satisfies the differential equation on I .

The above example illustrates that an equation $f(x, y) = 0$ might implicitly define more than one solution of a differential equation. Here are two further illustrations of implicitly defined solutions.

Example 1.5 Find solutions of the nonlinear differential equation

$$y \frac{dy}{dx} = -x.$$

Solution If we integrate both sides of the equation with respect to x , we obtain

$$\int y \frac{dy}{dx} dx = \int -x dx = -\frac{x^2}{2} + C.$$

Integration on the left is what might be called backwards implicit differentiation; find an expression which when differentiated with respect to x gives $y \frac{dy}{dx}$. Such an expression is $\frac{y^2}{2}$. In other words, we can write that

$$\frac{y^2}{2} = -\frac{x^2}{2} + C, \quad \text{or,} \quad x^2 + y^2 = 2C.$$

Since C is an arbitrary constant, but must be positive in order that the equation define functions, we replace it with D^2 , in which case, solutions of the differential equation are defined implicitly by

$$x^2 + y^2 = D^2.$$

Geometrically, this equation describes a family of circles of radius D centred at the origin, one of which is shown in Figure 1.2a. Algebraically, the equation defines

two continuous functions on the interval $-D \leq x \leq D$, $y(x) = \sqrt{D^2 - x^2}$ and $y(x) = -\sqrt{D^2 - x^2}$. Both are solutions of the differential equation on the interval $-D < x < D$. Notice that the endpoints $x = \pm D$ have been removed, and there are two reasons for this. Firstly, we agreed to discuss solutions of differential equations only on open intervals, and secondly, even if we had not made this agreement, these functions do not have derivatives at $x = \pm D$, and cannot therefore satisfy the differential equation at these points. The equation $x^2 + y^2 = D^2$ defines other functions on the interval $-D < x < D$, one being that in Figure 1.2b. But this function is not a solution of the differential equation on this interval because it does not have a derivative at $x = 0$. •

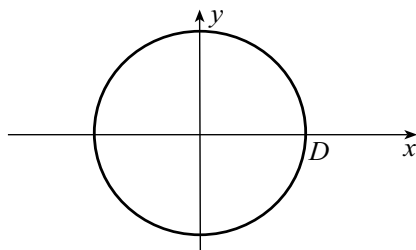


Figure 1.2a

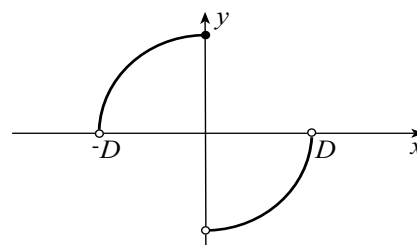


Figure 1.2b

Example 1.6 Show that the equation $y = (x + y)[\ln y + C]$, where C is a constant, implicitly defines solutions of the differential equation

$$\frac{dy}{dx} = -\frac{y^2}{x^2 + y^2 + xy}.$$

Solution Let $y(x)$ be any function that is implicitly defined by any one of the curves $y = (x + y)[\ln y + C]$. To show that its derivative satisfies the given differential equation, we use implicit differentiation. It is best (but not necessary) to isolate the constant C ,

$$\frac{y}{x + y} - \ln y = C.$$

Implicit differentiation now gives

$$\frac{(x + y)dy/dx - y(1 + dy/dx)}{(x + y)^2} - \frac{1}{y} \frac{dy}{dx} = 0.$$

When we multiply by $y(x + y)^2$, we get

$$y(x + y) \frac{dy}{dx} - y^2 \left(1 + \frac{dy}{dx} \right) - (x + y)^2 \frac{dy}{dx} = 0.$$

We now solve this for dy/dx ,

$$\frac{dy}{dx} = \frac{y^2}{y(x + y) - y^2 - (x + y)^2} = -\frac{y^2}{x^2 + xy + y^2}.$$

In Figure 1.3 we have shown plots of the curves $y = (x + y)[\ln y + C]$ for values $C = 1$ and $C = 2$. Any function defined by any one of these curves implicitly defines a solution of the differential equation on some interval.

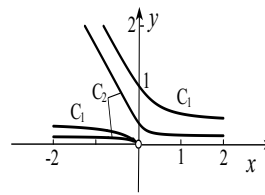


Figure 1.3

The interval could be $x < 0$, or $-\infty < x < \infty$, depending first on the choice of constant C , and then on the particular choice of curve resulting from that C . •

In these two examples, we used graphs to illustrate implicitly defined solutions of differential equations. To determine algebraically when an equation implicitly defines functions, we can use the “implicit function theorem” from calculus. It is discussed in Exercise 47 at the end of this section.

Boundary and Initial Conditions

When differential equations occur in applications, they are usually accompanied by subsidiary conditions called *initial* or *boundary conditions*. For example, suppose a drug such as glucose is administered intravenously into the bloodstream of a hospital patient at a constant rate R units per unit time. It is often assumed that the body uses the glucose up at a rate proportional to the amount present at that time. If $A(t)$ is the amount of glucose in the blood as a function of time t , then its derivative dA/dt is the time rate of change of the amount of glucose in the blood. It must be equal to the rate R at which glucose enters the bloodstream less the rate kA at which the body uses it up,

$$\begin{array}{ccc} \text{Rate of change} & & \text{Rate glucose} \\ \text{of the amount} & & \text{enters} \\ \text{of glucose} & & \text{is used up} \\ & \swarrow & \searrow \\ \frac{dA}{dt} = R - kA, & & \end{array} \quad (1.6)$$

where $k > 0$ is a constant. This is a differential equation for $A(t)$. If the initial amount of glucose in the blood at time $t = 0$ is A_0 , then $A(t)$ must also satisfy $A(0) = A_0$. This is an **initial condition** that solution(s) of the differential equation must also satisfy. In other words, the real problem is to find solution(s) of differential equation 1.6 that also satisfy the initial condition $A(0) = A_0$,

$$\frac{dA}{dt} = R - kA, \quad t > 0, \quad (1.7a)$$

$$A(0) = A_0. \quad (1.7b)$$

This is the form in which analysts in applied areas find differential equations — the differential equation is accompanied by subsidiary conditions that express other requirements of solutions. We call problem 1.7 an **initial-value problem**. It is not difficult to verify that solutions of the differential equation are

$$A(t) = \frac{R}{k} + Ce^{-kt}, \quad (1.8)$$

for any constant C . When we impose the initial condition, we obtain

$$A_0 = A(0) = \frac{R}{k} + C.$$

Thus, $C = A_0 - R/k$, and a solution of the initial-value problem is

$$A(t) = \frac{R}{k} + \left(A_0 - \frac{R}{k}\right)e^{-kt}. \quad (1.9)$$

This was your first exposure to mathematical modelling with differential equations. We turned the physical assumption that the body uses glucose up at a rate proportional to the amount present into differential equation 1.6. To complete the model we added the initial condition $A(0) = A_0$, resulting in initial-value problem 1.7.

Mathematical modelling is one of the most important areas of mathematics. It is used to describe oscillations of physical systems such as machinery, buildings during storms, and currents in electrical networks. Models have been constructed to describe the spread of diseases, the conduction of heat, and the diffusion of chemicals. The list of applications of mathematical models is vast, and ever growing. We will discuss mathematical modelling in numerous applications of differential equations throughout the rest of this text. We may not always mention that we are doing mathematical modelling, but the reader will recognize this.

A second situation that may already be familiar to some readers is that of a mass M suspended from a spring with constant k . If the mass is also subject to air drag that is proportional to its velocity, then displacement $x(t)$ of the mass from its equilibrium position as a function of time t must satisfy the differential equation

$$M \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0, \quad (1.10a)$$

where $\beta > 0$ is a constant. (Don't worry if you have never seen this equation before. We will derive and analyze it from every possible viewpoint in Chapter 5.) Accompanying the differential equation will be two initial conditions specifying the position and velocity of the mass at some initial time t_0 ,

$$x(t_0) = x_0, \quad x'(t_0) = v_0. \quad (1.10b)$$

The initial-value problem then consists of finding solution(s) of differential equation 1.10a that also satisfy initial conditions 1.10b.

Equation 1.1d, which is used to determine the deflection of a beam, is normally accompanied by four conditions, two at each end $x = 0$ and $x = L$ of the beam. Examples are

$$y(0) = y(L) = 0, \quad y''(0) = y''(L) = 0.$$

They are called **boundary conditions**. Differential equation 1.1d together with these boundary conditions is called a **boundary-value problem**. It is straightforward to verify that the function

$$y(x) = C_1 e^{kx} + C_2 e^{-kx} + C_3 \sin kx + C_4 \cos kx - \frac{1}{k^4}, \quad (1.11)$$

where C_1 , C_2 , C_3 , and C_4 are arbitrary constants, satisfies differential equation 1.1d. When the boundary conditions are applied, these constants must satisfy the four equations

$$\begin{aligned}
0 &= C_1 + C_2 + C_4 - 1/k^4, \\
0 &= C_1 e^{kL} + C_2 e^{-kL} + C_3 \sin kL + C_4 \cos kL - 1/k^4, \\
0 &= C_1 + C_2 - C_4, \\
0 &= C_1 e^{kL} + C_2 e^{-kL} - C_3 \sin kL - C_4 \cos kL.
\end{aligned}$$

The solution is

$$C_1 = \frac{1}{2k^4(1 + e^{kL})}, \quad C_2 = \frac{e^{kL}}{2k^4(1 + e^{kL})}, \quad C_3 = \frac{\csc kL - \cot kL}{2k^4}, \quad C_4 = \frac{1}{2k^4}.$$

With these, we have a solution of the boundary-value problem.

Example 1.7 Prove that $y = (C_1 + C_2 x)e^{-2x} + 1/4$ is a solution of the differential equation

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = 1$$

for all x . Find a solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$.

Solution When we substitute $y = (C_1 + C_2 x)e^{-2x} + 1/4$ into the left side of the differential equation, we get

$$\begin{aligned}
\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y &= (4C_1 - 4C_2 + 4C_2 x)e^{-2x} + 4(C_2 - 2C_1 - 2C_2 x)e^{-2x} \\
&\quad + 4[(C_1 + C_2 x)e^{-2x} + 1/4] = 1.
\end{aligned}$$

Because these calculations are valid for all x , the given function is a solution of the differential equation for all x . The initial conditions require

$$1 = y(0) = C_1 + \frac{1}{4}, \quad -1 = y'(0) = C_2 - 2C_1.$$

These imply that $C_1 = 3/4$ and $C_2 = 1/2$, and therefore a solution of the initial-value problem is $y = [(3 + 2x)e^{-2x} + 1]/4$. •

Example 1.8 Find a solution of the differential equation $(x - 1)\frac{dy}{dx} = 1$ that also satisfies the condition $y(-1) = 5$, and one that satisfies $y(4) = 6$.

Solution If we write

$$\frac{dy}{dx} = \frac{1}{x - 1}, \quad \text{integration gives} \quad y = \ln |x - 1| + C.$$

This compact notation from elementary calculus, although correct, disguises the true nature of the differential equation and its solutions. Clearly the differential equation makes no sense at $x = 1$. It should be considered on the intervals $x < 1$ and $x > 1$, or any subinterval of these intervals. Solutions on these intervals are

$$y = \begin{cases} \ln(1 - x) + C_1, & x < 1 \\ \ln(x - 1) + C_2, & x > 1 \end{cases}.$$

This is a much more accurate description of solutions than the compact notation $y = \ln |x - 1| + C$. To satisfy the condition $y(-1) = 5$, we use the solution $\ln(1 - x) + C_1$.

It requires $5 = \ln 2 + C_1$, from which $C_1 = 5 - \ln 2$. A solution is therefore $y(x) = \ln(1 - x) + 5 - \ln 2$, valid on the interval $x < 1$. To satisfy the condition $y(4) = 6$, we use the solution $\ln(x - 1) + C_2$. It requires $6 = \ln 3 + C_2$, from which $C_2 = 6 - \ln 3$. A solution is therefore $y(x) = \ln(x - 1) + 6 - \ln 3$, valid for $x > 1$. •

Did you notice that in Examples 1.7 and 1.8, we did not ask for “the” solution of the initial-value problem; we asked for “a” solution. The reason for this is that we have no way of knowing whether there is only one solution, or many solutions, and the word “the” would imply one solution. In Section 1.3, we find out when we can expect an initial-value problem to have one, and only one, solution. Until then, we must not be so bold as to say that once we have “a” solution to an initial-value problem, we have “the” solution to the problem.

Families of Solutions and Singular Solutions

We claim, without justification at the moment, that every solution of differential equation 1.6 can be written in form 1.8, and every solution of equation 1.1d can be expressed as 1.11. We call expression 1.8 a one-parameter family of solutions of equation 1.6, and expression 1.11 a four-parameter family of solutions of equation 1.1d. In both cases the number of parameters (or arbitrary constants) is the same as the order of the differential equation. We might suspect a general result emerging here to the effect that every solution of an n^{th} -order differential equation is contained in an n -parameter family of solutions. For linear differential equations this is true (a fact that will be proved later), but unfortunately it is not generally true for nonlinear equations. As an illustration, consider the nonlinear equation

$$\frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} \right)^2. \quad (1.12)$$

In Example 2.30, we apply standard techniques for solving differential equations to obtain the two-parameter family of solutions $y(x) = C_1 - \ln(C_2 + x)$. This two-parameter family of solutions does not, however, contain all solutions of the differential equation, because no choice of C_1 and C_2 will give the perfectly acceptable solution $y(x) = k$, where k is a constant. Constant functions are not particularly interesting, but they are nonetheless solutions that are not contained within the two-parameter family. Such solutions are called **singular solutions** of the two-parameter family. Notice that we say that a singular solution is a singular solution of the family of solutions, not a singular solution of the differential equation. We say this because sometimes by writing a family of solutions of a differential equation in a different form, the new family may contain the singular solution of the original family.

Example 1.9 Show that $y(x) = \frac{1}{1 + Ce^{-x}}$ is a solution of the nonlinear differential equation

$$\frac{dy}{dx} = y(1 - y)$$

on the interval $-\infty < x < \infty$ for any positive constant C . Can you see any singular solutions of this family of solutions?

Solution Since

$$\frac{dy}{dx} = \frac{-1}{(1 + Ce^{-x})^2}(-Ce^{-x}) = \frac{Ce^{-x}}{(1 + Ce^{-x})^2}$$

and

$$y(1 - y) = \frac{1}{1 + Ce^{-x}} \left(1 - \frac{1}{1 + Ce^{-x}} \right) = \frac{Ce^{-x}}{(1 + Ce^{-x})^2},$$

the function is indeed a solution of the differential equation. Because $C > 0$, the function is a solution on the interval $-\infty < x < \infty$. It is clear that the function $y(x) \equiv 0$ is also a solution of the differential equation, and it cannot be obtained from $(1 + Ce^{-x})^{-1}$ for any value of C . Thus, $y(x) \equiv 0$ is a singular solution of the one-parameter family of solutions.●

General Solutions

We have illustrated that a solution of a differential equation that contains the same number of arbitrary constants as the order of the differential equation may or may not contain all solutions of the differential equation. In spite of this unfortunate circumstance, there do exist classes of differential equations for which a solution with the same number of arbitrary constants as the order of the equation does represent all possible solutions (linear equations have this property). This prompts us to make the following definition.

Definition 1.5 An n -parameter family of solutions of an n^{th} -order differential equation is said to be a **general solution** if it contains all solutions of the differential equation[†].

Consequently, in order that a family of functions be a general solution of a differential equation, three conditions must be satisfied:

1. Each function in the family must be a solution of the differential equation.
2. The family must contain the requisite number of arbitrary constants (n for an n^{th} -order differential equation).
3. The family must contain all solutions of the differential equation.

Notice that we speak of “a” general solution of a differential equation, and not “the” general solution. The reason for this is that if there is one general solution of a differential equation, then there is an infinite number of general solutions, an infinite number of ways to express all solutions of the differential equation. For instance, two solutions of the linear differential equation $x^2y'' - 4xy' + 6y = 0$ are x^3 and x^2 . It is straightforward to show that for any constants C_1 and C_2 , the function $y(x) = C_1x^3 + C_2x^2$ is also a solution; it is a two-parameter family of solutions of the differential equation. We cannot yet, but in Chapter 4 we will be able to prove that every solution of the differential equation can be expressed in this form, and hence $y(x) = C_1x^3 + C_2x^2$ is a general solution. Functions $x^3 - x^2$ and $2x^3 + 4x^2$ are solutions of the differential equation, and $y(x) = C_3(x^3 - x^2) + C_4(2x^3 + 4x^2)$ is also a general solution. So also is $y(x) = C_5(3x^3 + 8x^2) + C_6(x^2 - 4x^3)$, but $y(x) = C_7(x^3 - 2x^2) + C_8(3x^3 - 6x^2)$ is not a general solution. Can you see the difference?

[†] Readers should be aware that not all authors agree on this definition of a general solution of a differential equation. Some do not require a general solution to contain all solutions of the differential equation. To them, an n -parameter family of solutions is a general solution.

Example 1.10 Show that $y(x) = e^x(C_1 \cos 2x + C_2 \sin 2x)$ is a two-parameter family of solutions of the linear differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

on the interval $-\infty < x < \infty$. Is it a general solution?

Solution If we substitute $y(x)$ into the left side of the differential equation,

$$\begin{aligned} \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y &= e^x(-3C_1 \cos 2x - 4C_1 \sin 2x - 3C_2 \sin 2x + 4C_2 \cos 2x) \\ &\quad - 2e^x(C_1 \cos 2x - 2C_1 \sin 2x + C_2 \sin 2x + 2C_2 \cos 2x) \\ &\quad + 5e^x(C_1 \cos 2x + C_2 \sin 2x) = 0. \end{aligned}$$

This shows that $y(x)$ satisfies the differential equation, and does so for all x . We cannot be sure that this two-parameter family of solutions contains all solutions of the differential equation, and cannot therefore claim that it is a general solution. In Chapter 4, when we consider the whole topic of linear differential equations, we rectify this.●

Example 1.11 Find a general solution for the differential equation $\frac{d^2y}{dx^2} = xe^{-x}$.

Solution Integration of both sides of the differential equation, with integration by parts on the right, gives

$$\frac{dy}{dx} = -xe^{-x} - e^{-x} + C_1.$$

A second integration yields

$$y(x) = xe^{-x} + 2e^{-x} + C_1x + C_2.$$

Elementary calculus assures us that inclusion of constants of integration leads to all solutions of the differential equation, and therefore we have a general solution.●

There is no simple procedure that always determines whether an n -parameter family of solutions of an n^{th} -order differential equation is a general solution. It may happen, as in Example 1.11, that the method of arriving at the n -parameter family of solutions guarantees that all solutions are captured. Unfortunately, this is the exception rather than the rule. In Example 1.7, we illustrated that $y = (C_1 + C_2x)e^{-2x} + 1/4$ is a two-parameter family of solutions of the linear differential equation $y'' + 4y' + 4y = 1$. At this time, it is not possible for us to show that all solutions of this differential equation can be obtained by specifying values for C_1 and C_2 , and hence we cannot claim to have a general solution.

In spite of the fact that we do not have a definitive procedure by which to determine whether an n -parameter family of solutions of an n^{th} -order differential equation is a general solution, there do exist classes of differential equations for which this is indeed the case. We shall certainly point them out as we encounter them. When this is the case, we need not worry about singular solutions; there cannot be any. As we have already mentioned, linear differential equations fall into this category, and we verify this in Section 2.1 and Chapter 4.

Some differential equations are immediately solvable (or, as we often say, immediately integrable). For example, to solve a linear differential equation of the form

$$\frac{dy}{dx} = M(x), \quad (1.13)$$

where $M(x)$ is a given function, we integrate both sides of the equation with respect to x to obtain a one-parameter family of solutions

$$y(x) = \int M(x) dx + C. \quad (1.14)$$

Once again, elementary calculus assures us that by including the constant of integration, we have all solutions of the differential equation. Hence, this one-parameter family of solutions is a general solution of the differential equation.

This result is easily extended to linear, n^{th} -order differential equations of the form

$$\frac{d^n y}{dx^n} = M(x), \quad n \text{ a positive integer.} \quad (1.15)$$

We integrate successively n times to obtain a general solution

$$y(x) = \int \cdots \int M(x) dx \cdots dx + C_1 + C_2 x + \cdots C_n x^{n-1}. \quad (1.16)$$

Example 1.12 Find the solution of the initial-value problem

$$\frac{d^2 y}{dx^2} = \cos 2x + x, \quad y(0) = 0, \quad y'(0) = 2.$$

Solution Two integrations of the differential equation give a general solution

$$y(x) = -\frac{1}{4} \cos 2x + \frac{x^3}{6} + C_1 x + C_2.$$

The initial conditions at $x = 0$ require C_1 and C_2 to satisfy the equations

$$0 = y(0) = -\frac{1}{4} + C_2, \quad 2 = y'(0) = C_1.$$

Thus, the solution of the initial-value problem is

$$y(x) = -\frac{1}{4} \cos 2x + \frac{x^3}{6} + 2x + \frac{1}{4}.$$

Notice that we called this function “the” solution of the initial-value problem rather than “a” solution. We are warranted in doing this because integration led to a general solution of the differential equation, meaning that the family contains all solutions of the differential equation. In addition, there is one, and only one, solution of the equations for C_1 and C_2 . Thus, the solution obtained can be the only solution of the problem.●

Particular Solutions

When a solution of a differential equation contains no arbitrary constants, it is called a **particular solution** of the differential equation. It follows therefore that particular solutions can be obtained by assigning specific values to arbitrary constants in a family of solutions. For example, $y(x) = 5 - \ln(3 + x)$ is a particular solution of differential equation 1.12, as is $y(x) = -\ln x$, both being obtained from $y(x) = C_1 - \ln(C_2 + x)$ by specifying values for C_1 and C_2 . On the other hand, the singular solution $y(x) = 10$ is also a solution, but it cannot be obtained from this two-parameter family of solutions. The function $y(x) \equiv 0$ is also a particular solution of equation 1.12, and it is a singular solution. It is called the **trivial** solution. We are invariably interested in nontrivial solutions of differential equations, but it may be important to note that a differential equation has the trivial solution.

Example 1.13 Find a particular solution of the differential equation

$$5\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2y = 4.$$

Solution In Chapter 4 we develop systematic techniques for finding particular solutions for differential equations such as this. But clearly such techniques are not needed here; a simple glance tells us that $y(x) = 2$ is a solution, and since it contains no arbitrary constants, it is a particular solution.●

We have five objectives in studying differential equations. One objective is to determine conditions that guarantee that differential equations have solutions. A second objective is to find solutions of differential equations, and we will discuss many techniques for finding solutions. Thirdly, if we cannot find solutions of a particular differential equation, we would like to glean whatever information is available about solutions. Fourthly, we want to know what properties solutions of differential equation can be expected to possess. Finally, we want to use differential equations to model a multitude of applications.

The fourth objective is especially important when it comes to applications. When differential equations arise in applications they often involve parameters (constants) of the application. We want to know how values of these parameters affect solutions. For instance differential equation 1.1e describes displacement of a mass m on the end of a spring (with constant k) subject to drag proportional to velocity (with constant β). Intuitively, values of the parameters m , β , and k determine the nature of the motion of the mass. Solutions of the differential equation should reflect these expectations. In other words, we want to understand every aspect of differential equations; conditions guaranteeing existence of solutions, techniques for finding solutions, information about solutions when they are unattainable, properties of solutions, and applications.

EXERCISES 1.1

In Exercises 1–8 determine open intervals in which each function of the family satisfies the differential equation. State whether the differential equation is linear or nonlinear.

1. $y(x) = 2 + Ce^{-x^2}$; $\frac{dy}{dx} + 2xy = 4x$

2. $y(x) = \frac{x^3}{2} + Cx^3 e^{1/x^2}; \quad x^3 \frac{dy}{dx} + (2 - 3x^2)y = x^3$
3. $y(x) = C_1 \sin 3x + C_2 \cos 3x; \quad \frac{d^2 y}{dx^2} + 9y = 0$
4. $y(x) = \frac{C_1^2 e^{2x} + 1}{2C_1 e^x} + C_2; \quad \left(\frac{d^2 y}{dx^2} \right)^2 = 1 + \left(\frac{dy}{dx} \right)^2$
5. $y(x) = C_1 e^{2x} \cos(x/\sqrt{2}) + C_2 e^{2x} \sin(x/\sqrt{2}); \quad 2 \frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 9y = 0$
6. $y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 \cos x + C_4 \sin x; \quad \frac{d^4 y}{dx^4} + 5 \frac{d^2 y}{dx^2} + 4y = 0$
7. $y(x) = (C_1 + C_2 x - x^2/4)e^{4x}; \quad 2 \frac{d^2 y}{dx^2} - 16 \frac{dy}{dx} + 32y = -e^{4x}$
8. $y(x) = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x) + 1/4; \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4y = 1$

In Exercises 9–18 determine whether the differential equation is linear or nonlinear. Make no attempt to solve the equation.

9. $2x \frac{d^2 y}{dx^2} + x^3 y = x^2 + 5$
10. $2x \frac{d^2 y}{dx^2} + x^3 y = x^2 + 5y$
11. $2x \frac{d^2 y}{dx^2} + x^3 y = x^2 + 5y^2$
12. $x \frac{d^3 y}{dx^3} + 3x \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 1 - \sin x$
13. $x \frac{d^3 y}{dx^3} + 3y \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 10 \sin x$
14. $\sin y \frac{d^3 y}{dx^3} + 3x \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 10 \sin x$
15. $y'' - 3y' - 2y = 9 \sec^2 x$
16. $yy'' + 3y' - 2y = e^x$
17. $(1 + y')^{1/3} + x^2 = 4$
18. $y'''' + y'' - y = \ln x$

In Exercises 19–22 find a particular solution of the differential equation in Exercise 3 that satisfies the conditions.

19. $y(0) = 1, \quad y'(0) = 6$
20. $y(0) = 2, \quad y(\pi/2) = 3$
21. $y(\pi/12) = 0, \quad y'(\pi/12) = 1$
22. $y(1) = 1, \quad y(2) = 2$

In Exercises 23–27 find a general solution for the differential equation.

23. $\frac{dy}{dx} = 6x^2 + 2x$
24. $\frac{dy}{dx} = \frac{1}{9 + x^2}$
25. $\frac{d^2 y}{dx^2} = 2x + e^x$
26. $\frac{d^2 y}{dx^2} = x \ln x$
27. $\frac{d^3 y}{dx^3} = \frac{1}{3x^5}$
28. (a) Find a solution $y = y(x)$ of the initial-value problem

$$\frac{dy}{dx} = \frac{x}{(1 - x^2)^{1/3}}, \quad y(0) = 2.$$

- (b) What is the largest interval on which $y(x)$ is a solution of the initial-value problem?
- (c) What is the largest interval on which $y(x)$ is defined?

29. (a) Verify that $y = x^4/4$ is a solution of the initial-value problem

$$\frac{dy}{dx} = 2x\sqrt{y}, \quad y(0) = 0.$$

- (b) Find another solution of the problem.

In Exercises 30–34 show that the equation implicitly defines solutions of the differential equation. State whether the differential equation is linear or nonlinear.

30. $4x^2 + Cy^2 = 1$, $\frac{dy}{dx} = \frac{4xy}{4x^2 - 1}$ Do this by isolating C before differentiation, and by not isolating it.

31. $y^2 = x^3/(C - x)$, $\frac{dy}{dx} = \frac{y^3 + 3x^2y}{2x^3}$

32. $y - x + \sin^{-1}(x + y) = C$, $\frac{dy}{dx} = \frac{\sqrt{1 - (x + y)^2} - 1}{\sqrt{1 - (x + y)^2} + 1}$

33. $x^2y^2 + y \ln(2 + \sin x) = C$, $2xy^2 + \frac{y \cos x}{2 + \sin x} + [2x^2y + \ln(2 + \sin x)] \frac{dy}{dx} = 0$

34. $x^3 + y^2 - \ln|y + x^2| = C$, $\left(\frac{1}{y + x^2} - 2y\right) \frac{dy}{dx} = 3x^2 - \frac{2x}{y + x^2}$

35. A curve in the one-parameter family $x^3 + y^3 = Cxy$ is called a **folium of Descartes**.

- (a) Show that the equation implicitly defines solutions of the differential equation

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

- (b) Find an implicit definition for the solution of the differential equation that satisfies the condition $y(1) = 1$. Plot the curve defined by this equation.

- (c) Determine the interval on which the function defined implicitly in part (b) is a solution of the initial-value problem.

36. An important consideration in Section 4.4 will be to determine values of the constant m so that e^{mx} is a solution of a linear differential equation such as

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0.$$

Find two such values.

37. An important consideration in Section 4.10 will be to determine values of the constant m so that x^m is a solution of a linear differential equation such as

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0.$$

Find two such values.

38. (a) Jason initially at O (figure to the right) walks along the edge of a swimming pool (the y -axis) towing his sailboat by a string of length L . If the boat starts at Q and the string always remains straight, show that the equation of the curved path $y = y(x)$ followed by the boat must satisfy the initial-value problem

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}, \quad y(L) = 0.$$

- (b) Find a solution of the initial-value problem.

39. (a) Show that $y(x) = 1 - (x^3 + C)^{-1}$ is a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 3x^2(y - 1)^2.$$

- (b) Find a singular solution.

- (c) Repeat parts (a) and (b) if the differential equation is given in the form

$$\frac{1}{(y - 1)^2} \frac{dy}{dx} = 3x^2.$$

40. (a) Show that $y(x) = Ce^{2x}$ is a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 2y.$$

- (b) Show that there is a particular solution that passes through any given point (x_0, y_0) , and that this solution can be obtained by specifying C appropriately.

41. (a) Verify that a one-parameter family of solutions for the differential equation

$$2x \frac{dy}{dx} = y$$

is defined implicitly by the equation $y^2 = Cx$.

- (b) Show that, with the exception of points on the y -axis, there is a particular solution that passes through any given point (x_0, y_0) , and that this solution can be obtained by specifying C appropriately.

42. Consider the differential equation $dy/dx = 1/x^2$.

- (a) Find a solution that satisfies the condition $y(1) = 1$.

- (b) Find a solution that satisfies the condition $y(-1) = 2$.

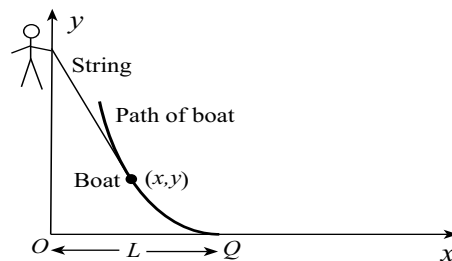
- (c) Is there a solution that satisfies both the conditions in parts (a) and (b)?

43. Determine whether the following functions are solutions of the differential equation $dy/dx = -x/y$ on the interval $-2 < x < 2$. If your answer is no, explain why not.

(a) $y(x) = \sqrt{4 - x^2}$

(b) $y(x) = -\sqrt{4 - x^2}$

(c) $y(x) = \begin{cases} \sqrt{4 - x^2}, & -2 < x < 0 \\ -\sqrt{4 - x^2}, & 0 \leq x < 2 \end{cases}$



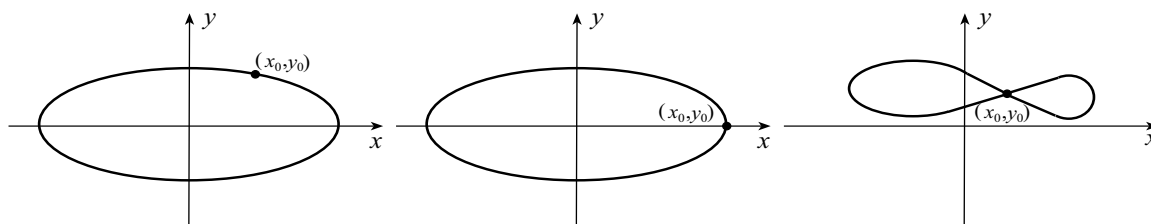
44. Determine whether the following functions are solutions of the differential equation $x(dy/dx) - 2y = 0$ on the interval $-\infty < x < \infty$. If your answer is no, explain why not.
- (a) $y(x) = 2x^2$
 - (b) $y(x) = -3x^2$
 - (c) $y(x) = \begin{cases} -3x^2, & x < 0 \\ 2x^2, & x \geq 0 \end{cases}$
45. (a) Show that $y(x) = Cx^2$ is a one-parameter family of solutions of the differential equation in Exercise 44.
- (b) If the differential equation is expressed in the form $\frac{dy}{dx} = \frac{2y}{x}$, are the functions in the one-parameter family $y(x) = Cx^2$ still solutions?
46. Results of this section would seem to suggest that differential equations always have an infinite number of solutions. Can you find a differential equation that has exactly one solution? Can you find a differential equation that has no solutions?
47. In this exercise, we discuss the “implicit function theorem” from calculus. It states that: An equation $F(x, y) = 0$ defines a differentiable function $y(x)$ in some open interval $|x - x_0| < \delta$ around x_0 if x_0 and $y_0 = y(x_0)$ satisfy $F(x_0, y_0) = 0$ and $\frac{\partial F(x_0, y_0)}{\partial y} \neq 0$. A proof of this result can be found in advanced calculus books. We illustrate its graphical interpretation here. Suppose the curve defined by the equation $F(x, y) = 0$ is as shown in the left figure below. It is clear that the curve defines a function in an interval around x_0 . Its derivative can be found by implicit differentiation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0.$$

When this is solved for dy/dx at $x = x_0$, the derivative is

$$\left. \frac{dy}{dx} \right|_{x_0} = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}.$$

Notice that the requirement $F_y(x_0, y_0) \neq 0$ of the theorem ensures that the denominator of this expression does not vanish. The curves in the middle and right figures do not define y as a function of x in an open interval around x_0 , let alone differentiable ones. For the middle figure, $F_y(x_0, y_0)$ would be equal to zero, and for the right figure $F_y(x_0, y_0)$ would not exist.



- (a) Show that $F(x, y) = x^2 + y^2 - 4$ is an example of the left figure above provided that $-2 < x < 2$.
- (b) Show that $F(x, y) = x^2 + y^2 - 4$ is an example of the middle figure when $x_0 = \pm 2$.
- (c) Show that $F(x, y) = y^2 + x^2(x^2 - 1)$ is an example of the right figure when $x_0 = 0$.
- (d) Show that $F(x, y) = y^3 - x$ with $x_0 = 0$ defines a function in an interval around x_0 , but not a differentiable one.

1.2 Qualitative Discussions of First-order Differential Equations and Direction Fields

First-order differential equations in y as a function of x can be expressed symbolically in the form

$$F(x, y, y') = 0, \quad (1.17)$$

where F is some function of x , $y(x)$, and $y'(x)$. Many first-order equations can also be written in the form

$$\frac{dy}{dx} = f(x, y). \quad (1.18)$$

For example, we can solve the differential equation

$$x^2y + (x + 1)\frac{dy}{dx} - e^x = 0,$$

for dy/dx , and write

$$\frac{dy}{dx} = \frac{e^x - x^2y}{x + 1}.$$

Depending on the form of $f(x, y)$, various techniques can be developed to find solutions of differential equation 1.18, and this is the purpose of Chapter 2. Given an arbitrary function $f(x, y)$, however, there is a distinct possibility that none of these techniques will work. Fortunately, even though we may not always be able to find solutions of the differential equation, it may be possible to find a great deal of information about them. In this section we discuss algebraic and geometric techniques that enable us to discover properties of solutions of differential equation 1.18 prior to making any attempt to find solutions. Such properties can also serve as checks if, and when, we actually solve the differential equation.

Algebraic Considerations

Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 + 5.$$

The fact that dy/dx is always positive tells us that no solution to the differential equation can have a relative maximum or a relative minimum; all solutions are increasing functions. Suppose we concentrate on a solution that also satisfies the initial condition $y(0) = 0$. The differential equation shows that the first derivative of such a solution at $x = 0$ is $y'(0) = 5$ (Figure 1.4). We can show that the graph of a solution through $(0, 0)$ must have a point of inflection at $(0, 0)$. We use the differential equation to calculate the second derivative of solutions,

$$\frac{d^2y}{dx^2} = 2x + 2y\frac{dy}{dx}.$$

Clearly this is equal to zero at $(0, 0)$. To show that this is a point of inflection, we must show that the second derivative

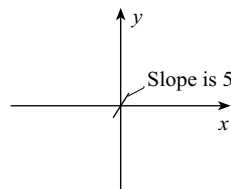


Figure 1.4

changes sign as x increases through zero. Figure 1.4 shows that when x is less than zero, so also is y . Since $dy/dx > 0$, it follows that $d^2y/dx^2 < 0$ when x is less than zero. Similar reasoning shows that $d^2y/dx^2 > 0$ when x is larger than zero. This confirms that any solution of the differential equation that passes through the point $(0, 0)$ must have a point of inflection at the point.

Here is a second example.

Example 1.14 Show that any solution curve of the differential equation

$$\frac{dy}{dx} = x^3(2 + \sin y)$$

that crosses the y -axis has a relative minimum at the crossing point.

Solution Certainly any solution curve crossing the y -axis must have slope zero there since $x = 0$. Furthermore, since $2 + \sin y$ is always positive, it follows that dy/dx changes from a negative quantity to a positive quantity as x increases through zero. Consequently, $x = 0$ must yield a relative minimum on every solution curve. (Solution curves are plotted in Exercise 33.)●

Geometric Considerations

Because the derivative of a function is the slope of the tangent line to its graph, first-order differential equations can be approached geometrically. For example, when a function satisfies the differential equation

$$\frac{dy}{dx} = \frac{x}{y}, \tag{1.19}$$

the slope of its graph at any point (x, y) must be the x -coordinate of the point divided by the y -coordinate of the point. For instance, if a function satisfies equation 1.19, and its graph passes through the point $(1, 2)$, then the slope of its graph must be $1/2$ at the point. If the graph passes through the point $(-3, 4)$, it must have slope $-3/4$ at this point. In other words, the differential equation associates a slope with every point in the xy -plane (except points on the x -axis where $y = 0$), and for a function to satisfy the differential equation and pass through a particular point (x, y) , its slope there must be x/y .

What this suggests is that at each point in the xy -plane, we draw a short line segment with slope equal to that defined by the differential equation. Certainly we cannot draw line segments at every point in the xy -plane, but we could draw them at a predetermined grid of points. This was done for differential equation 1.19, and the result is shown in Figure 1.5. Line segments were drawn at the points (m, n) where m and $n \neq 0$ are integers satisfying $-4 \leq m, n \leq 4$. This array of line segments is called the **direction field** for differential equation 1.19. It would perhaps be more appropriate to call Figure 1.5 “a” direction field rather than “the” direction field because it is hardly unique. The values of m and n can be extended beyond ± 4 , and line segments can also be drawn at noninteger values. We feel that it is easier to call figures like 1.5 the direction field, and we hope that readers are not offended by this slight misuse of language.

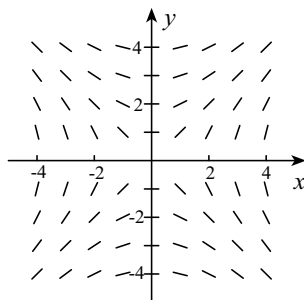


Figure 1.5

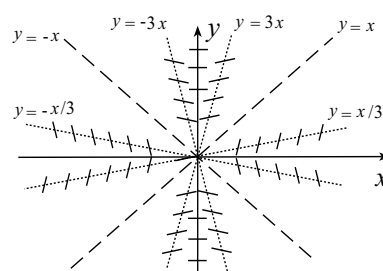


Figure 1.6

Needless to say, a computer could be programmed to draw line segments at many more points than those shown in Figure 1.5, but even without a computer, we can quickly develop a reasonable picture of the direction field in this example, and others. If we set

$$\frac{dy}{dx} = \frac{x}{y} = C,$$

where C is a constant, we obtain the straight line $y = x/C$. This curve is called an **isocline**. It has the property that at every point on the isocline, the line segment of the direction field has slope C . For example, when $C = 1/2$, the isocline is $y = 2x$ and the slope of line segments at each point on $y = 2x$ is $1/2$. At each point on the isocline $x/y = -3$, or, $y = -x/3$, the slope of the line segments is -3 . We have shown various isoclines in Figure 1.6 complete with line segments. At each point on a given isocline, all line segments have the same slope. Thus, by drawing a few isoclines associated with a differential equation, it may be possible to obtain a fairly clear picture of the direction field. If the isoclines in Figure 1.6 are deleted, but the line segments are left, a direction field similar to that in Figure 1.5 is obtained.

Direction fields can provide information about solutions of a differential equation even if it is difficult, or impossible, to find solutions algebraically. The field guides solutions through the xy -plane. To see this, suppose we want a solution of differential equation 1.19 that passes through the point $(1, 3)$. A solution is represented geometrically by a curve through $(1, 3)$ that always remains tangent to the segments in the direction field. One such curve is shown in Figure 1.7a. We have also shown a solution curve that passes through the point $(-1, -2)$. Thus, even though we have not solved differential equation 1.19, we have a fairly clear picture of solutions through the points $(1, 3)$ and $(-1, -2)$. In addition, we would be led to believe that these are the only curves through the points $(1, 3)$ and $(-1, -2)$ that follow the direction field. In other words, the direction field is suggesting that through any point (except for points on the x -axis), there is one, and only one, solution of the differential equation.

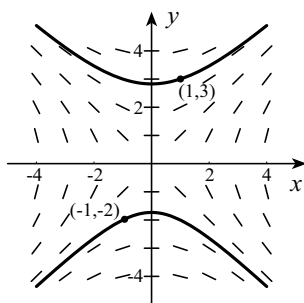


Figure 1.7a

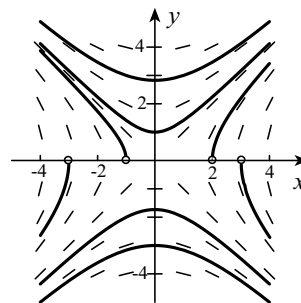


Figure 1.7b

We have shown more solutions in Figure 1.7b. They appear to be hyperbolas asymptotic to the lines $y = \pm x$. (This is indeed the case; we will show later that $x^2 - y^2 = C$ is a 1-parameter family of solutions of the differential equation.)

Isoclines and direction fields do not give algebraic representations for solutions of differential equations; they give geometric visualizations. But even this limited information on the nature of solutions can be useful in applied problems. As a second example, consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2. \quad (1.20)$$

Innocent as this differential equation appears, it cannot be solved in terms of elementary functions, but by drawing its direction field, we obtain an excellent visualization of its solutions. Isoclines are circles $x^2 + y^2 = C$. At each point on any such circle, the slope of the line segment in the direction field is the square of its radius. We have shown a number of isoclines with their associated line segments in Figure 1.8a. Figure 1.8b shows a computer generated direction field and illustrates a typical solution curve, in spite of the fact that it is not possible to find an equation for the curve in terms of elementary functions. The figure also suggests that through any point (x_0, y_0) in the xy -plane, there is one and only one solution of differential equation 1.20. In other words, the direction field associated with equation 1.20, and only the direction field, leads us to believe that the equation has a unique solution through any given point in the xy -plane.

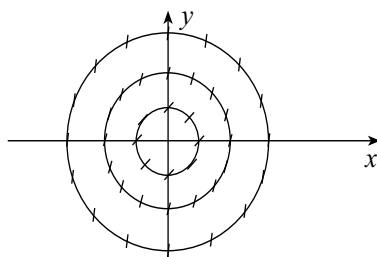


Figure 1.8a

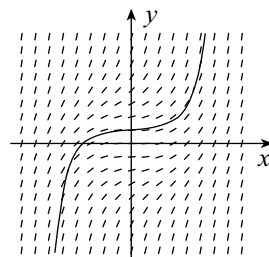


Figure 1.8b

Example 1.15 The direction field for the differential equation $2x(dy/dx) = y$ is shown in Figure 1.9a. Draw some curves that adhere to the tangent line segments. Does it appear that through any point (x_0, y_0) , there is exactly one solution of the differential equation?

Solution Curves that remain tangent to the direction field in Figure 1.9a are shown in Figure 1.9b. They appear to be parabolas. The x -axis with equation $y(x) = 0$ is also a solution. Through any point (x_0, y_0) , except points on the y -axis, there seems to be exactly one solution of the differential equation. The fact that line segments cannot be drawn at points on the y -axis suggests that finding solutions of the differential equation that pass through these points will be a problem. We shall see this in Section 1.3 when we discuss existence and uniqueness of solutions to first-order differential equations. •

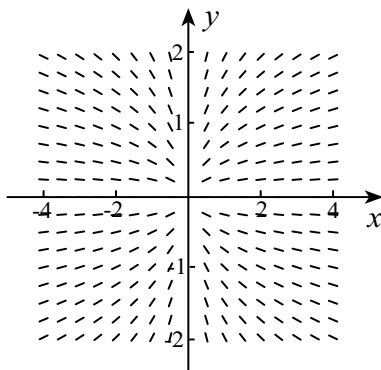


Figure 1.9a

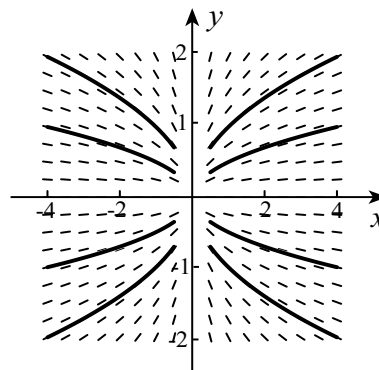


Figure 1.9b

Example 1.16 Initial-value problem 1.7 describes the amount of glucose in the blood when glucose is added at a constant rate R units per unit time and the body uses it up at a rate proportional to how much is present (with an initial amount A_0 at time $t = 0$). We saw that a solution of the problem is $A(t) = R/k + (A_0 - R/k)e^{-kt}$. Graphs of this function for various initial amounts A_0 are shown in Figure 1.10. Clearly all solutions are asymptotic to the line $A = R/k$. When $A_0 < R/k$, solutions approach the asymptote from below, and when $A_0 > R/k$, the approach is from above. In other words, no matter what the initial amount of glucose in the body, the amount always levels off at R/k as time progresses. Show that the same information can be obtained from the direction field associated with the differential equation.

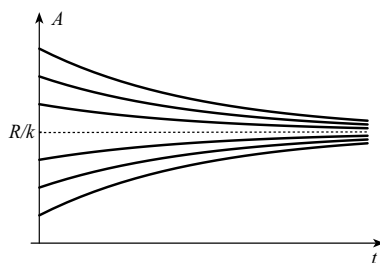


Figure 1.10

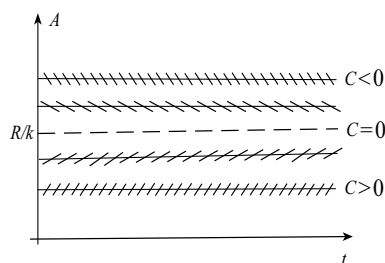


Figure 1.11

Solution Because of unspecified parameters k and R in the differential equation, we cannot use a computer to draw the direction field. We resort to isoclines; they

are horizontal lines $R - kA = C$, or $A = (R - C)/k$. The slope of line segments along an isocline is C . When $C = 0$, the isocline is the horizontal line $A = R/k$ where line segments have slope zero (Figure 1.11). When $C > 0$, the isocline is a horizontal line below the previous isocline and line segments have positive slopes. When $C < 0$, the isocline is a horizontal line above $A = R/k$, and line segments have negative slopes. The closer the value of C to zero, the closer the isocline is to $A = R/k$, and the closer slopes of line segments are to zero. All of this is shown in Figure 1.11. This direction field makes it clear that all solutions that start on the A -axis are asymptotic to the line $A = R/k$. Thus, the direction field has shown that solution curves of the initial-value problem must be exactly as pictured in Figure 1.10. •

Direction fields can sometimes fail to guide solutions of a differential equation at certain points. An example is the differential equation

$$\frac{dy}{dx} = 3y^{2/3}.$$

The direction field for $x \geq 0$ is shown in Figure 1.12a, where it is important to notice that line segments have zero slope along the x -axis. It is straightforward to verify that a one-parameter family of solutions of the differential equation is $y = (x + C)^3$. To find a function in this family whose graph passes through the point $(1, 0)$, we set $0 = (1 + C)^3$, from which $C = -1$. Hence, a solution of the differential equation that passes through the point $(1, 0)$ is $y = (x - 1)^3$. It is shown in Figure 1.12b. But the function $y(x) \equiv 0$ also satisfies the differential equation and passes through the point $(1, 0)$. In other words, the direction field fails to guide a solution that reaches the point $(1, 0)$. It could move along the curve $y = (x - 1)^3$ or it could move along the x -axis. The same situation occurs at every point on the x -axis; $y(x) \equiv 0$ is a solution of the differential equation passing through the point $(a, 0)$ as is $y(x) = (x - a)^3$. Notice that $y(x) = 0$ is a singular solution of the family $y = (x - a)^3$.

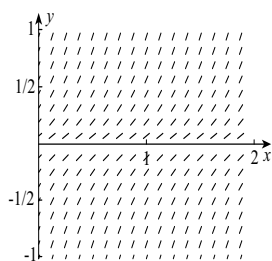


Figure 1.12a

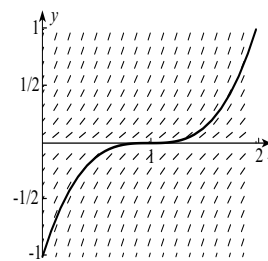


Figure 1.12b

EXERCISES 1.2

1. Show that no solution curve to the differential equation

$$\frac{dy}{dx} = 2 + x^2 y^2$$

can have a point of inflection in the first quadrant.

2. Show that a solution curve to the initial-value problem

$$\frac{dy}{dx} = \frac{y(x+y)}{1+3x^2y^2}, \quad y(1) = -1,$$

has a relative maximum at the point $(1, -1)$.

3. Show that any solution curve of the initial-value problem

$$\frac{dy}{dx} = ay - by^2, \quad y(0) = y_0,$$

where a and b are constants, and $0 < y_0 < a/(2b)$, has a point of inflection when $y = a/(2b)$.

4. Show that if a solution curve to the differential equation

$$\frac{dy}{dx} + y = x$$

intersects the line $y = x$, then the curve has a relative minimum at the point of intersection.

In Exercises 5–14 identify isoclines for the differential equation.

- | | |
|----------------------------------|-------------------------------------|
| 5. $\frac{dy}{dx} = x - 2$ | 6. $\frac{dy}{dx} = y + 4$ |
| 7. $\frac{dy}{dx} = 3x - 2y$ | 8. $\frac{dy}{dx} = y + 5x$ |
| 9. $\frac{dy}{dx} = 2x^2 + 3y^2$ | 10. $\frac{dy}{dx} = x^2 - 4y^2$ |
| 11. $\frac{dy}{dx} = x^3y$ | 12. $\frac{dy}{dx} = \frac{x}{y^3}$ |
| 13. $\frac{dy}{dx} = (x^2 + 1)y$ | 14. $\frac{dy}{dx} = ye^{2x}$ |

In Exercises 15–22 draw the direction field associated with the differential equation.

- | | |
|--------------------------------------|---------------------------------------|
| 15. $\frac{dy}{dx} = 2y$ | 16. $\frac{dy}{dx} = y + 1$ |
| 17. $\frac{dy}{dx} = -\frac{2x}{3y}$ | 18. $\frac{dy}{dx} = xy$ |
| 19. $\frac{dy}{dx} = x + y$ | 20. $\frac{dy}{dx} = 4x^2 + y^2$ |
| 21. $\frac{dy}{dx} = \frac{y}{3x}$ | 22. $\frac{dy}{dx} = -\frac{4x^3}{y}$ |

23. (a) Show that a solution of the differential equation

$$\frac{dy}{dx} = 1 - xy^2, \quad x > 0,$$

that passes through the point $(1, 1)$ must have a relative maximum at this point.

- (b) Show that every solution of the differential equation that intersects the curve $y = 1/\sqrt{x}$ has a relative maximum at the point of intersection.
- (c) Plot the direction field for the differential equation to corroborate parts (a) and (b).
24. (a) Show that isoclines for the differential equation $\frac{dy}{dx} = \frac{3x^2}{1-y}$ are parabolas.
- (b) Draw the direction field.

25. (a) Show that isoclines for the differential equation

$$\frac{dy}{dx} = -\frac{y(1-x)}{x(1-y)}$$

have equations of the form $y = \frac{Cx}{(C+1)x-1}$. Do they all pass through the point $(1, 1)$?

- (b) Draw isoclines in the first quadrant corresponding to values $C = -2, -1, 1, 2$. What isocline results if we let $C \rightarrow \infty$?
- (c) Draw the direction field in the first quadrant.
- (d) Do curves that remain tangent to line segments in the direction field appear to be closed curves around the point $(1, 1)$?
26. (a) Find equations for, and describe, isoclines for the differential equation

$$\frac{dy}{dx} = y(4-y).$$

- (b) Draw the direction field in the first quadrant. What is the maximum slope of line segments?
- (c) Sketch solution curves that begin at points $y = y_0$ on the y -axis. Do they differ for $0 < y_0 < 2$, $2 < y_0 < 4$, and $y_0 > 4$?
- (d) Verify that solutions for which $0 < y_0 < 2$ have points of inflection. What are values of y at points of inflection?
27. Consider the differential equation

$$\frac{dy}{dx} = (4-y)(y-2), \quad x > 0.$$

A student once gave the following reasoning to show that a solution curve of the differential equation that meets with the line $y = 2$ must have a relative minimum at the point. Slope of the curve at the point is $dy/dx = 0$. Furthermore, when $y < 2$, the slope is negative, and when $y > 2$, the slope is positive. Since the first derivative changes from a negative quantity to a positive quantity, there must be a relative minimum at the point.

- (a) Show that the direction field negates the above reasoning. Where did the reasoning go wrong?
- (b) Which solution curves to the differential equation have points of inflection?
28. Show that solution curves of the initial-value problem

$$\frac{dy}{dx} = (c-y)(d-y), \quad y(x_0) = y_0 < c,$$

where $d > c > 0$, cannot have a point of inflection.

29. Show that if a solution curve $y = y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} = y$$

intersects the x -axis at some value x_0 , and $y'(x_0) \neq 0$, then the curve cannot intersect the x -axis at a second point.

30. By examining the direction field in Exercise 16 find a constant function satisfying the differential equation.

31. By examining the direction field in Exercise 19 find a linear function satisfying the differential equation.
32. (a) Draw the direction field for the differential equation

$$\frac{dy}{dx} = \sin y,$$

- in the region $0 < x < 6$, $-\pi \leq y \leq \pi$.
- (b) What can you say about a solution with initial value $y(x_0) = 0$, where $x_0 > 0$?
- (c) What can you say about any solution with initial value $y(x_0) = y_0$, where $x_0 > 0$ and $0 < y_0 < \pi$?
- (d) What can you say about a solution with initial value $y(x_0) = y_0$, where $x_0 > 0$ and $-\pi < y_0 < 0$?
- (e) What can you say about solutions with initial values $y(x_0) = \pm\pi$, where $x_0 > 0$?
33. (a) Verify that $\tan y - \sec y = x^4/4 + C$ implicitly defines solutions of the differential equation in Example 1.14.
- (b) Plot the curves for $C = -0.2, -0.5, -1, -2$ to illustrate that each curve has a relative minimum at $x = 0$.

34. Consider the initial-value problem

$$y \frac{dy}{dx} = x + 1, \quad y(0) = 1.$$

- (a) Show that every solution satisfies $y'(0) \neq 0$ and $y''(0) = 0$. Can you conclude that every solution curve has a point of inflection at the point $(0, 1)$?
- (b) Plot the direction field for the differential equation and use it to decide whether a solution curve has a point of inflection at $(0, 1)$.
35. Consider the initial-value problem

$$\frac{dy}{dx} = x^3 \sin y, \quad y(0) = c,$$

where c is a given constant.

- (a) Show that when $0 < c < \pi$, a solution curve has a relative minimum at $(0, c)$.
- (b) Show that when $-\pi < c < 0$, a solution curve has a relative maximum at $(0, c)$.
- (c) Can you determine whether $(0, 0)$ is a maximum or a minimum in the case that $c = 0$? Plot the direction field to decide the issue.
36. (a) Verify that the equation $y^3 - 4y - x^2 + 4x = C$, where C is a constant, implicitly defines solutions of the differential equation

$$\frac{dy}{dx} = \frac{2x - 4}{3y^2 - 4}.$$

It is probably fair to say that we get very little information about the nature of solutions from this implicit definition. There are two things that we could do in order to visualize solutions. This we do in parts (b) and (c).

- (b) Draw the direction field associated with the differential equation on the rectangle $-4 \leq x \leq 8$, $-6 \leq y \leq 6$. Sketch a few curves that follow the direction field.
- (c) Use a computer to plot the curves $y^3 - 4y - x^2 + 4x = C$ for $C = -4, -2, 0, 2, 4, 6$.

1.3 Existence and Uniqueness

Before attempting to solve a differential equation, it is natural to ask whether the equation even has solutions; or perhaps more appropriately, on what interval(s) do solutions of the differential equation exist, if any. Otherwise, considerable time could be wasted searching for nonexistent solutions. This would be important for an applied mathematician who has modelled some physical situation with a differential equation. If there is no solution to the differential equation, the model is problematic; it demands too much, or perhaps demands the impossible.

When differential equations arise in applications, they are accompanied by initial or boundary conditions, and these conditions usually determine values for integration constants. The result is one and only one solution of the differential equation and subsidiary conditions. But is this always the case, or could it happen that a differential equation, together with what are felt to be appropriate subsidiary conditions, has more than one solution? Once again, this would be an important question for the mathematical modeler; it would be undesirable to have more than one solution to a model that describes a physical problem; if it does, the model does not demand enough. In this section we give conditions that ensure **existence** and **uniqueness** of solutions to initial-value problems.

First-order Differential Equations

We begin with first-order differential equations that are expressible in the form

$$\frac{dy}{dx} = f(x, y). \quad (1.21)$$

This is sometimes called the **standard form** for first-order differential equations. The following example illustrates how initial conditions influence whether an initial-value problem has a solution and whether it is unique.

Example 1.17 Use isoclines to draw the direction field associated with the differential equation $x \frac{dy}{dx} - y = 0$. Does your drawing suggest the following:

- (a) the differential equation has at least one solution when subjected to the initial condition $y(x_0) = y_0$ when $x_0 \neq 0$;
- (b) the differential equation has no solution when subjected to the initial condition $y(0) = y_0 \neq 0$;
- (c) the differential equation has an infinity of solutions when subjected to the initial condition $y(0) = 0$.

Solution Isoclines are straight lines $y = Cx$ so that the slope of the line segment along an isocline is the slope of the isocline itself. We have shown line segments in Figure 1.13. (a) Through any point not on the y -axis, there appears to be a unique solution of the differential equation, namely the isocline through that point. (b) There certainly seems to be a problem with points on the y -axis since line segments cannot be drawn at such points.

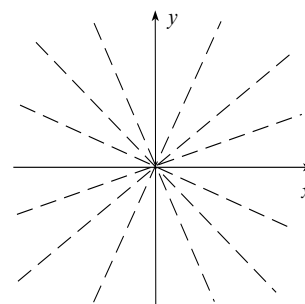


Figure 1.13

(c) There appears to be an infinity of solutions through the origin, namely, the isoclines $y = Cx$.•

This example indicates that initial (and/or boundary) conditions for a differential equation play a fundamental role in whether a problem has a solution, and whether the solution is unique. The following theorem gives conditions that guarantee that differential equation 1.21 subject to the initial condition $y(x_0) = y_0$ has a unique solution.

Theorem 1.1 Suppose that $f(x, y)$ and its partial derivative $\partial f/\partial y$ are continuous in an open rectangle R : $a < x < b$, $c < y < d$. If (x_0, y_0) is a point in R , then there exists an open interval containing x_0 in which the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \quad (1.22)$$

has a unique continuously differentiable solution[†].

We have shown the situation in Figure 1.14. To solve problem 1.22 is to find a curve through the point (x_0, y_0) whose slope at each point (x, y) is $f(x, y)$. Theorem 1.1 guarantees that one and only one solution curve exists when (x_0, y_0) is in R . The theorem makes no prediction as to the length of the interval. It may be as large as $a < x < b$; it may be shorter (as in Figure 1.14).

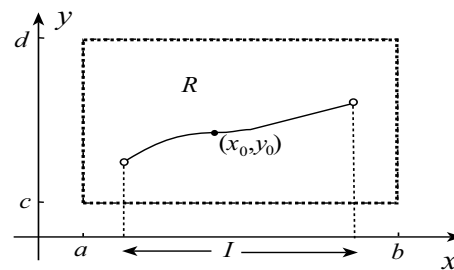


Figure 1.14

It is worthwhile pointing out that if the condition that $\partial f/\partial y$ be continuous in Theorem 1.1 is removed, it is possible to prove that solutions to initial-value problem 1.22 exist, but uniqueness is not guaranteed.

It is an immediate consequence of this theorem that two solution curves of the differential equation in 1.22 cannot intersect inside the rectangle of Figure 1.14. If the coordinates of the point of intersection were (x_0, y_0) , then there would be two different solutions satisfying initial value problem 1.22, contradicting the uniqueness of the theorem. It also follows that solution curves completely fill the rectangle $a < x < b$, $c < y < d$; through each point of the rectangle, there is a unique solution curve of the differential equation.

A solution of differential equation 1.21 on an interval I is a function that satisfies 1.21 at each point in I . It necessarily has a derivative at each point at which $f(x, y)$ is defined. When $f(x, y)$ satisfies the continuity condition of Theorem 1.1, it follows that a solution of the differential equation must have a continuous first derivative on I . Its graph cannot have a corner.

As mentioned in the footnote, the conditions of Theorem 1.1 are sufficient to guarantee existence of a unique solution of problem 1.22; they are not necessary. In

[†] The conditions on $f(x, y)$ can be weakened, but the ones stated here suffice for our purposes. A proof of the theorem with less stringent conditions can be found in Appendix F.

other words, a problem might have a unique solution when $f(x, y)$ and $\partial f/\partial y$ are not continuous in a rectangle containing the point (x_0, y_0) . Alternatively, it might have more than one solution, and it might have no solution. The theorem simply does not apply.

The differential equation of Example 1.17 is an excellent example to illustrate these ideas. When we write the differential equation in the form

$$\frac{dy}{dx} = \frac{y}{x},$$

we see that $f(x, y) = y/x$ and $\partial f/\partial y = 1/x$ are continuous in any rectangle not containing points on the y -axis. Hence, when the initial condition $y(x_0) = y_0$ of part (a) is specified at $x_0 \neq 0$, we can draw a rectangle R around point (x_0, y_0) in which f and $\partial f/\partial y$ are continuous. If $x_0 > 0$, the largest rectangle R satisfying Theorem 1.1 is the right half-plane $x > 0$ (Figure 1.15a); if $x_0 < 0$, we can take R as the left half-plane $x < 0$ (Figure 1.15b).

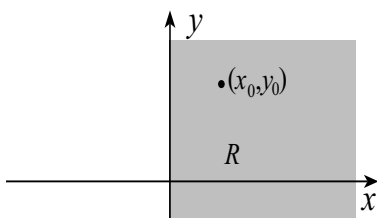


Figure 1.15a

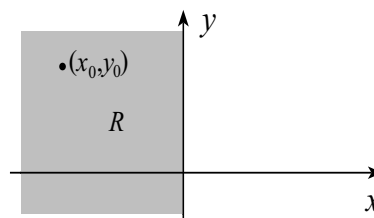


Figure 1.15b

In Figure 1.15a, Theorem 1.1 guarantees a unique solution valid in some open interval I contained in $0 < x < \infty$. In Figure 1.15b, a unique solution is known to exist in some open interval I contained in $-\infty < x < 0$. It is straightforward to verify that a solution through the point (x_0, y_0) is the straight line $y = y_0 x/x_0$, and it is defined throughout the whole semi-infinite interval.

When the initial condition $y(0) = y_0$ is specified at $x = 0$ as in parts (b) and (c) of the example, it is impossible to draw a region R containing the point $(0, y_0)$ in which $f(x, y) = y/x$ and $\partial f/\partial y = 1/x$ are continuous. Does Theorem 1.1 imply that there does not exist a unique solution through $(0, y_0)$? No!! The theorem does not apply.

Notice also that in part (a) of the example, $y = y_0 x/x_0$ is a solution of the problem in its given form $x dy/dx = y$, $y(x_0) = y_0$ for all x . It is not a solution just on some interval contained in $x < 0$ or in $x > 0$ as predicted by Theorem 1.1; it is a solution on $-\infty < x < \infty$. Theorem 1.1 cannot predict this. The theorem requires us to write the differential equation in the form $dy/dx = y/x$, and once in this form, it is necessary to distinguish between positive and negative values of x .

Usually, we solve initial-value problem 1.22 by finding a one-parameter family of solutions of the differential equation, and then use the initial condition to find the value of the parameter. Graphically, the one-parameter family of solutions is a one-parameter family of curves. Graphing these curves can be straightforward, or sufficiently difficult as to require a graphing calculator or computer. A simple situation is for equations of form 1.13. This is illustrated in the next example.

Example 1.18 Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 3x^2 - 2x.$$

Draw solution curves and find the one that also satisfies the condition $y(2) = 12$.

Solution Integration of both sides of the differential equation with respect to x gives the one-parameter family of solutions

$$y(x) = x^3 - x^2 + C.$$

Graphically, solution curves are the one-parameter family of cubic curves in Figure 1.16. Constant C represents a vertical shift of one curve relative to another. When we require the solution to satisfy the initial condition $y(2) = 12$, constant C must have value 8. Geometrically, the initial condition picks that curve from the family that passes through the point $(2, 12)$. It is the thickest curve in the figure. •

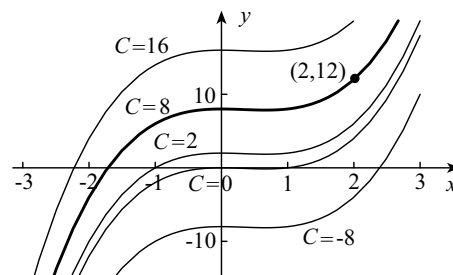


Figure 1.16

The simplicity of this example is the exception rather than the rule. Usually, the constant of integration does not appear simply as an additive constant in an explicit representation of a one-parameter family of solutions for the differential equation. Here is an example to illustrate.

Example 1.19 (a) Verify that a one-parameter family of solutions of the differential equation

$$\frac{dy}{dx} = -\frac{3y^2 + 1}{6y^2 + 6xy}$$

is defined implicitly by the equation $2y^3 + 3xy^2 + x = C$.

- (b) Plot solution curves and find the function that satisfies the condition $y(2) = 1$.
 (c) Along which curves are line segments in the direction field associated with the differential equation not defined?
 (d) Is there an asymptote for solution curves of the differential equation?

Solution (a) Implicit differentiation of $2y^3 + 3xy^2 + x = C$ gives

$$6y^2 \frac{dy}{dx} + 3y^2 + 6xy \frac{dy}{dx} + 1 = 0, \quad \text{from which} \quad \frac{dy}{dx} = -\frac{3y^2 + 1}{6y^2 + 6xy}.$$

Thus, functions defined implicitly by the equation $2y^3 + 3xy^2 + x = C$ do satisfy the differential equation.

(b) A few solution curves are plotted in Figure 1.17. To obtain the solution satisfying $y(2) = 1$, we substitute these values into the one-parameter family of curves, $2(1)^3 + 3(2)(1)^2 + 2 = C$. Thus, $C = 10$, and this identifies the thickest curve in Figure 1.17. That part of this curve above the x -axis defines the solution of the differential equation that also satisfies $y(2) = 1$.

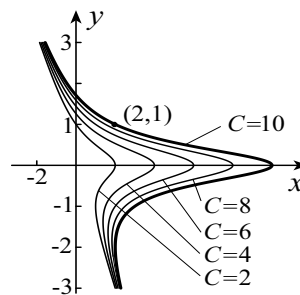


Figure 1.17

(c) Line segments cannot be drawn at any points (x, y) at which $6y^2 + 6xy = 6y(y + x) = 0$; that is, along the x -axis and the line $y = -x$.

(d) Solution curves in Figure 1.17 suggest an asymptote but do not make its equation clear. We can find it by writing solutions in the form

$$x = \frac{C - 2y^3}{3y^2 + 1}.$$

For large absolute values of y , x can be approximated by $-2y/3$; that is, the asymptote is $x = -2y/3$ or $y = -3x/2$.•

Did you notice the change in language in Examples 1.18 and 1.19 compared to that in Section 1.1. In Section 1.1, we asked for “a” solution to an initial-value problem because we had no idea whether there would be one solution or many solutions. In Examples 1.18 and 1.19, we asked for “the” solution. In both examples, the differential equation satisfies the conditions of Theorem 1.1, (although we did not check this), and hence there is a unique solution to the problem. We can therefore ask for “the” solution to the problem, and once we have “a” solution, we have “the” solution.

We cannot predict where a solution of initial-value problem 1.22 will encounter difficulties simply by examining the function $f(x, y)$. This is illustrated in the following example.

Example 1.20 Solve the initial-value problem $\frac{1}{y^2} \frac{dy}{dx} = 2x, \quad y(0) = 1$.

Solution If we integrate both sides of the differential equation with respect to x , we obtain

$$-\frac{1}{y} = x^2 + C.$$

The initial condition requires $-1 = C$. The solution of the initial-value problem is therefore defined implicitly by

$$-\frac{1}{y} = x^2 - 1 \quad \implies \quad y(x) = \frac{1}{1 - x^2}.$$

It is the solution, but only on the interval $-1 < x < 1$, because the function becomes unbounded at $x = \pm 1$. When we write the differential equation in the form $dy/dx = 2xy^2$, the function $2xy^2$ satisfies Theorem 1.1 for all x and y , yet the solution breaks down at $x = \pm 1$.

It would be natural to ask whether the direction field for this differential equation indicates the asymptotic nature of the solution. The direction field is shown in Figure 1.18, and it does not suggest an asymptote. Slopes of line segments get large quite quickly as y increases, but there is nothing to suggest an asymptote. This example is discussed further in Exercise 23.•

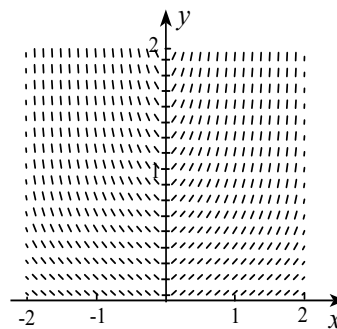


Figure 1.18

The differential equation in this example is nonlinear, and what we saw is generally the situation for nonlinear equations. We cannot predict the domain of validity of solutions of nonlinear problems based on where $f(x, y)$ and $\partial f/\partial y$ are continuous. When the differential equation is linear, however, the domain of the solution is exactly the same as the interval on which $f(x, y)$ and $\partial f/\partial y$ are continuous. We shall see this in Section 2.1.

It is perhaps worth re-emphasizing that our discussions have been confined to differential equations that can be expressed in form 1.21. Such first-order equations are said to be of **first-degree**. The following equation cannot be put in form 1.21, and it is not therefore first-degree

$$\left(\frac{dy}{dx}\right)^4 - x^2y\left(\frac{dy}{dx}\right)^3 - 2x^4y = 0.$$

Existence and Uniqueness for n^{th} -order Differential Equations

We now extend Theorem 1.1 to initial-value problems associated with n^{th} -order differential equations. Differential equations of order n in a function $y(x)$ can always be expressed in the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \quad (1.23)$$

where F is some function of its arguments. Any of the arguments $x, y, y', \dots, y^{(n-1)}$ can be absent, but $y^{(n)}$ must be present (else the differential equation is not of order n). We consider only equations that can be solved uniquely for $y^{(n)}$, and therefore that the differential equation can be written in the form

$$y^{(n)}(x) = f(x, y, y', \dots, y^{(n-1)}), \quad (1.24)$$

where f is a function of $x, y, y', \dots, y^{(n-1)}$. This eliminates from discussion a differential equation such as

$$\left(\frac{d^2y}{dx^2}\right)^3 + (2+x)y\frac{d^2y}{dx^2} + 2xy^2 = 0.$$

This differential equation cannot be solved for d^2y/dx^2 .

Because the function $f(x, y)$ in Theorem 1.1 is only a function of x and y , it is assumed that f and $\partial f/\partial y$ are continuous in some rectangle R : $a < x < b$, $c < y < d$ of the xy -plane. Function $f(x, y, y', y'', \dots, y^{(n-1)})$ in equation 1.24 is a function of n variables, and therefore the rectangle is replaced by a region R in the n -dimensional space of $x, y, y', y'', \dots, y^{(n-1)}$,

$$R : a < x < b, \quad c < y < d, \quad h_3 < y' < k_3, \quad h_4 < y'' < k_4, \quad \dots \quad h_{n+1} < y^{(n-1)} < k_{n+1}.$$

In place of function f and $\partial f/\partial y$ being continuous in Theorem 1.1, there will be $f(x, y, y', y'', \dots, y^{(n-1)})$ and its n partial derivatives with respect to $y, y', \dots, y^{(n-1)}$. In other words, we have the following theorem.

Theorem 1.2 Consider the initial-value problem of differential equation 1.24 subject to the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad \dots \dots y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

Suppose that $f(x, y, y', y'', \dots, y^{(n-1)})$ and its n partial derivatives with respect to $y, y', \dots, y^{(n-1)}$ are continuous in a region

$$R : a < x < b, \quad c < y < d, \quad h_3 < y' < k_3, \quad h_4 < y'' < k_4, \quad \dots \quad h_{n+1} < y^{(n-1)} < k_{n+1}.$$

If $(x_0, y_0, y'_0, \dots, y_0^{(n-1)})$ is a “point” in R , then there is an open interval containing x_0 in which the initial-value problem has a unique solution. The solution and its first $n - 1$ derivatives will be continuous in the interval.

Theorems 1.1 and 1.2 provide a way that can sometimes be used to verify that an n -parameter family of solutions of an n^{th} -order differential equation is a general solution to the differential equation. Here are two examples.

Example 1.21 Show that $y(x) = Ce^{-2x} + 2x - 1$ is a general solution of the differential equation

$$\frac{dy}{dx} = 4x - 2y, \quad -\infty < x < \infty.$$

Solution If we substitute the one-parameter family of functions into the differential equation, we get

$$\frac{dy}{dx} + 2y - 4x = (-2Ce^{-2x} + 2) + 2(Ce^{-2x} + 2x - 1) - 4x = 0,$$

confirming that each function in the family is a solution of the differential equation. There is one parameter in the family, and therefore to prove that the family is a general solution, we must show that every solution of the differential equation can be written in the form $Ce^{-2x} + 2x - 1$. Let then, $y_1(x)$ be any solution of the differential equation, and suppose that the value of this solution at some value $x = x_0$ is $y_1(x_0) = a$. According to Theorem 1.1, $y_1(x)$ is the unique solution of the initial-value problem

$$\frac{dy}{dx} = 4x - 2y, \quad y(x_0) = a.$$

We can write $y_1(x)$ in the form $Ce^{-2x} + 2x - 1$ if there exists a constant C such that

$$a = Ce^{-2x_0} + 2x_0 - 1.$$

But this equation always has solution $C = (a - 2x_0 + 1)e^{2x_0}$. Thus, every solution of the differential equation can be expressed in the form $Ce^{-2x} + 2x - 1$, and this one-parameter family of solutions is a general solution of the differential equation. •

Example 1.22 Show that $y(x) = C_1 \cos 2x + C_2 \sin 2x$ is a general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0, \quad -\infty < x < \infty.$$

Solution It is straightforward to confirm that each function in the two-parameter family is a solution of the differential equation, just substitute to see. There are two parameters in the family, and therefore to prove that the family is a general

solution, we must show that every solution of the differential equation can be written in the form $C_1 \cos 2x + C_2 \sin 2x$. Let then, $y_1(x)$ be any solution of the differential equation, and suppose that values of this solution and its first derivative at some value $x = x_0$ are $y_1(x_0) = a$ and $y'_1(x_0) = b$. According to Theorem 1.2, $y_1(x)$ is the unique solution of the initial-value problem

$$\frac{d^2y}{dx^2} + 4y = 0, \quad y(x_0) = a, \quad y'(x_0) = b.$$

We can write $y_1(x)$ in the form $C_1 \cos 2x + C_2 \sin 2x$ if there exist constants C_1 and C_2 such that

$$a = C_1 \cos 2x_0 + C_2 \sin 2x_0, \quad b = -2C_1 \sin 2x_0 + 2C_2 \cos 2x_0.$$

But these equations always have a solution, namely,

$$C_1 = a \cos 2x_0 - \frac{b}{2} \sin 2x_0, \quad C_2 = \frac{b}{2} \cos 2x_0 + a \sin 2x_0.$$

Thus, every solution of the differential equation can be expressed in the form $C_1 \cos 2x + C_2 \sin 2x$, and this two-parameter family of solutions is a general solution of the differential equation. •

You may have noticed that the differential equations in these two examples are linear. In Chapter 4, we will prove this in general; every n -parameter family of solutions of a linear n^{th} -order differential equation is a general solution of the differential equation. This is not necessarily true for nonlinear differential equations, however; an n -parameter family of solutions of a nonlinear n^{th} -order differential equation may, or may not, be a general solution of the differential equation. It is correspondingly more difficult to use Theorems 1.1 and 1.2 to show that an n -parameter family of solutions of a nonlinear differential equation is a general solution of the differential equation. In fact, it may be impossible because the family may not be a general solution.

Example 1.23 Determine whether the one-parameter family of functions $y(x) = -1/(x + C)$ is a general solution of the nonlinear differential equation

$$\frac{dy}{dx} = y^2.$$

Solution Each function in the family is a solution of the differential equation as is confirmed by the calculations

$$\frac{dy}{dx} = \frac{1}{(x + C)^2}, \quad \text{and} \quad y^2 = \frac{1}{(x + C)^2}.$$

To prove that the family is a general solution, we must show that every solution of the differential equation can be written in the form $-1/(x + C)$. Let then, $y_1(x)$ be any solution of the differential equation, and suppose that the value of this solution at some value $x = x_0$ is $y_1(x_0) = a$. According to Theorem 1.1, $y_1(x)$ is the unique solution of the initial-value problem

$$\frac{dy}{dx} = y^2, \quad y(x_0) = a.$$

We can write $y_1(x)$ in the form $-1/(x + C)$ if there exists a constant C such that

$$a = \frac{-1}{x_0 + C} \quad \implies \quad C = -\frac{1}{a} - x_0.$$

Since C does not exist when $a = 0$, solutions of the differential equation for which $y(x_0) = 0$ cannot be expressed in the form $-1/(x + C)$. (For instance, the trivial function $y(x) \equiv 0$ is a solution of the differential equation that cannot be expressed in the form $-1/(x + C)$.) Thus, the one-parameter family of solutions $y = -1/(x + C)$ is not a general solution of the differential equation. •

EXERCISES 1.3

1. Can $y(x) = x^{3/2}$ be a solution of a differential equation satisfying the conditions of Theorem 1.1 on an open rectangle containing the origin? Explain.
2. Can $y(x) = x^{1/3}$ be a solution of a differential equation satisfying the conditions of Theorem 1.1 on an open rectangle containing the origin? Explain.
3. Suppose $y_1(x)$ and $y_2(x)$ are two solutions of differential equation 1.22 satisfying the conditions of Theorem 1.1 in the rectangle of Figure 1.14. Can these solutions intersect inside the rectangle?
4. Consider the initial-value problem

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0.$$

- (a) Show that the conditions of Theorem 1.1 are satisfied in the entire xy -plane.
 - (b) Verify that $y(x) = \tan x$ satisfies the problem. On what interval is it a solution?
5. (a) Consider the initial-value problem

$$x \frac{dy}{dx} = 2y, \quad y(1) = 2.$$

- Verify that $y(x) = Cx^2$ is a one-parameter family of solutions of the differential equation.
- (b) What is the solution of the initial-value problem? What is its domain?
 - (c) What does Theorem 1.1 predict for the domain of the solution? Explain the difference.
 - (d) Is the solution in part (b) the only solution of the initial value problem on the interval $-\infty < x < \infty$?
6. (a) In Exercise 31 of Section 1.1, you verified that the equation $y^2 = x^3/(C - x)$ implicitly defines solutions of the differential equation

$$2x^3 \frac{dy}{dx} = y^3 + 3x^2 y.$$

- Find the explicit solution satisfying the initial condition $y(1) = 1$. What is its domain?
- (b) What does Theorem 1.1 predict for an interval of validity of the solution in part (a)?

In Exercises 7–10 find a region of the xy -plane in which Theorem 1.1 guarantees a unique solution of the differential equation passing through a point in the region.

- | | |
|--|--|
| 7. $\frac{dy}{dx} = \sqrt{xy}$ | 8. $x \frac{dy}{dx} = \ln(y - 2)$ |
| 9. $\frac{dy}{dx} = \frac{e^y}{\sqrt{1 - 4x^2}}$ | 10. $x^2 \frac{dy}{dx} = \sqrt{y - 1}$ |

In Exercises 11–14 find a region of xyy' -space in which Theorem 1.2 guarantees a unique solution of the differential equation passing through a point in the region.

11. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 5\ln x$ 12. $e^x \frac{d^2y}{dx^2} - 2y\sqrt{\frac{dy}{dx}} = e^x$
 13. $\frac{d^2y}{dx^2} + \sqrt{\frac{dy}{dx}} - 1 - (x+5)\ln y = \sqrt{1-x^2}$ 14. $(y^2+1)\frac{dy}{dx}\frac{d^2y}{dx^2} = \sqrt{x-1}$

In Exercises 15–18 find a one-parameter family of solutions of the differential equation. Plot or draw the family of curves, and determine which solution curve satisfies the initial condition.

15. $\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(0) = 2$ 16. $x^2 \frac{dy}{dx} = 1, \quad y(1) = 1$
 17. $\frac{dy}{dx} = xe^{-2x}, \quad y(0) = 3/4$ 18. $x \frac{dy}{dx} = 3\sqrt{x^4+x^6}, \quad y(0) = 4$

In Exercises 19–21 verify that the one-parameter family of functions, defined implicitly by the equation, satisfies the differential equation. Plot or draw the family of curves, and determine which solution curve satisfies the initial condition.

19. $4x^2 + 9y^2 = C, \quad 9y\frac{dy}{dx} = -4x, \quad y(1) = 1$
 20. $x^2(x^2 - 2y^2) = C, \quad \frac{dy}{dx} = \frac{x^2 - y^2}{xy}, \quad y(-2) = -1$
 21. $x^3y^3 + 3xy + 5x^2 = C, \quad \frac{dy}{dx} = -\frac{10x + 3x^2y^3 + 3y}{3x^3y^2 + 3x}, \quad y(2) = 2$
 22. If function $f(x, y)$ and its first partial derivative with respect to y in differential equation 1.21 are continuous in the circle $x^2 + y^2 < R^2$, is there a unique solution to the differential equation through any point (x_0, y_0) in the circle?
 23. Solve the differential equation in Example 1.20 subject to the initial condition $y(0) = a$, where $a > 0$ is a constant. Demonstrate that the domain of the solution is $-1/\sqrt{a} < x < 1/\sqrt{a}$. In other words, the domain can be made as small as desired, and as large as desired, simply by choosing a appropriately. This confirms the remark that we made following Example 1.20. Theorems 1.1 and 1.2 cannot predict the domain of validity of solutions to initial-value problems.
 24. Prove that the solution to the following initial-value problem is always positive

$$\frac{dy}{dx} = y^2 \cos^4 x, \quad y(0) = 1.$$

25. Prove that $y(x) = x^3 + C/x$ is a general solution of the differential equation

$$x \frac{dy}{dx} + y = 4x^3$$

on the interval $x > 0$.

26. Prove that $y(x) = C_1 \cos x + C_2 \sin x + x$ is a general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = x, \quad -\infty < x < \infty.$$

- 27.** Determine whether the one-parameter family of functions $y(x) = 1/(1 + Ce^x)$ is a general solution of the nonlinear differential equation

$$\frac{dy}{dx} = y(y - 1).$$

Summary of Chapter 1

Because it is important to have a firm grasp on terminology associated with solutions to differential equations, we began our studies by describing various types of solutions. Whether a function is defined explicitly or implicitly, it is a solution of a differential equation on an open interval I if substitution of the function into the differential equation reduces it to an identity on the interval.

1. A particular solution is a solution that contains no arbitrary constants.
2. An n -parameter family of solutions of an n^{th} -order differential equation is a solution that contains n arbitrary (and independent) constants.
3. A general solution of an n^{th} -order differential equation is an n -parameter family of solutions that contains every solution of the differential equation; that is, every solution can be obtained by specifying values for the parameters.
4. A singular solution of a family of solutions to a differential equation is a solution of the equation that is not contained in the family; that is, no values of the parameters yield the solution. Notice once again that a singular solution is a singular solution of a family of solutions, not of the differential equation itself.

It is often possible to obtain algebraic and geometric information about solutions to first-order differential equations even before any attempt is made to find solutions. Particularly useful are direction fields. At every point (x, y) at which the function $f(x, y)$ in differential equation $dy/dx = f(x, y)$ is defined, it associates a line segment with slope $f(x, y)$. These line segments graphically guide solution curves through the xy -plane. They can predict behaviours of solutions, and also provide checks on the validity of solutions.

Knowing that a particular differential equation under study has solutions is important. It would certainly be deflating to spend hours trying to find a solution, only to have someone come along and say that the differential equation has no solution. Theorems 1.1 and 1.2 provide conditions that guarantee that initial-value problems associated with first- and n^{th} -order differential equations have unique solutions. It is worth emphasizing that these theorems do not apply to boundary-value problems. For instance, they cannot be used to make predictions pertaining to the existence and uniqueness of solutions to the following boundary-value problem on the interval $0 < x < 3$.

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y^2 = x, \quad y(0) = 2, \quad y(3) = -2.$$

Review Exercises For Chapter 1

In Exercises 1–2 determine open intervals in which each function of the family satisfies the differential equation. State whether the differential equation is linear or nonlinear.

1. $y(x) = C_1 \frac{\sin x}{\sqrt{x}} + C_2 \frac{\cos x}{\sqrt{x}}; \quad x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1/4)y = 0$

2. $y(x) = \frac{x}{1 + Cx}; \quad x^2 \frac{dy}{dx} = y^2$

3. Does the solution curve to the initial-value problem

$$\frac{dy}{dx} = \sin y, \quad y(0) = \pi/2$$

have a point of inflection at the point $(0, \pi/2)$?

4. (a) Show that $y(x) = C_1x^2 + C_2/x^2 + x^3/5$ is a two-parameter family of solutions of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y = x^3.$$

(b) Find the solution that also satisfies the initial conditions $y(1) = 0$ and $y'(1) = 2$.

5. (a) Find the solution $y = y(x)$ of the initial-value problem

$$\frac{dy}{dx} = \frac{x^2}{(1-x^3)^{3/2}}, \quad y(0) = 5/3.$$

(b) What is the largest interval on which $y(x)$ is the solution of the problem?

In Exercises 6–7 show that the equation implicitly defines solutions of the differential equation.

6. $x^2y^3 + 2x + 3y = C, \quad \frac{dy}{dx} = -\frac{2xy^3 + 2}{3x^2y^2 + 3}$

7. $4x = y(\ln y + C)^2, \quad x > 0, \quad \frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$

8. (a) Show that $y(x) = -(x^2 + C)^{-1}$ is a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 2xy^2.$$

(b) Find a singular solution.

(c) Show that there is a solution passing through any given point (x_0, y_0) in the xy -plane, but the solution cannot always be obtained by specifying C .

9. Determine whether the following functions are solutions of the differential equation $dy/dx = -2x/(3y)$ on the interval $-3 < x < 3$. If your answer is no, explain why not.

(a) $y(x) = \sqrt{6 - 2x^2/3}$

(b) $y(x) = -\sqrt{6 - 2x^2/3}$

(c) $y(x) = \begin{cases} -\sqrt{6 - 2x^2/3}, & -3 < x < 0 \\ \sqrt{6 - 2x^2/3}, & 0 \leq x < 3 \end{cases}$

In Exercises 10–13 identify isoclines for the differential equation.

10. $\frac{dy}{dx} = 10x - 3y$

11. $\frac{dy}{dx} = y^2 + 4x$

12. $\frac{dy}{dx} = 3x^2 + 2y^2$

13. $\frac{dy}{dx} = xy + 4x^2y$

In Exercises 14–15 draw the direction field associated with the differential equation.

14. $\frac{dy}{dx} = x - y$

15. $\frac{dy}{dx} = -\frac{x}{y}$

16. (a) Draw the direction field for the differential equation

$$\frac{dy}{dx} = y(a - y),$$

where $a > 0$ is a constant, for $y \geq 0$.

- (b) If the initial condition $y(x_0) = y_0$ is attached to the differential equation, for which values of $y_0 > 0$ do solution curves have a point of inflection?
17. Repeat Exercise 29 in Section 1.2 for the differential equation

$$\frac{d^2y}{dx^2} = f(x)y,$$

when $f(x)$ is a function that takes on only positive values.

18. By examining the direction field in Exercise 14 find a linear function satisfying the differential equation.
19. (a) What is the isocline corresponding to slope 1 for the differential equation

$$x \frac{dy}{dx} = y + \sqrt{x^2 + y^2}?$$

- (b) What is the isocline for slope -1 ?
- (c) Show that isoclines are straight line segments.
20. (a) Draw that part of the direction field in the first quadrant for the differential equation

$$\frac{dy}{dx} = y - x^2.$$

- (b) What are slopes of line segments along the parabola $y = x^2$?
- (c) Does it appear that every solution curve passing through a point on the parabola $y = x^2$ has a relative maximum there? Prove this algebraically.
21. Check that the trivial function $y(x) \equiv 0$ satisfies the differential equation

$$x \frac{dy}{dx} + 5y = 0$$

for $-\infty < x < \infty$. Does this contradict Theorem 1.1?

22. Can $y(x) = (x-1)^{2/5}$ be a solution of a differential equation satisfying the conditions of Theorem 1.1 on an open rectangle containing the point $(1, 0)$? Explain.
23. Consider the initial-value problem

$$\frac{dy}{dx} = 1 + 4y^2, \quad y(0) = 0.$$

- (a) Show that the conditions of Theorem 1.1 are satisfied in the entire xy -plane.
- (b) Verify that $y(x) = (1/2) \tan 2x$ satisfies the problem. On what interval is it a solution?
24. (a) Consider the initial-value problem

$$x^2 \frac{dy}{dx} = y, \quad y(1) = 2.$$

Verify that $y(x) = Ce^{-1/x}$ is a one-parameter family of solutions of the differential equation.

- (b) What is the solution of the initial-value problem? What is its domain?

25. (a) Draw the direction field for the differential equation

$$\frac{dy}{dx} = \sin(x + y)$$

for $-4 \leq x \leq 4$, $-4 \leq y \leq 4$.

- (b) Does the function $y(x) = -x - \pi/2$ appear to follow the direction field? Verify algebraically that the function is indeed a solution of the differential equation.
- (c) Are there any other straight line solutions?
26. If function $f(x, y)$ and its first partial derivative with respect to y in differential equation 1.21 are continuous in the ellipse $2x^2 + 3y^2 < 35$, is there a unique solution to the differential equation through any point (x_0, y_0) in the ellipse?
27. Prove that the solution to the initial-value problem

$$\frac{dy}{dx} = (y^2 - 1) \cos(x^2 + y^2), \quad y(1) = 1/2$$

satisfies $|y(x)| < 1$ for all x in its domain of definition.

28. Prove that $y(x) = (C + e^x)/x^2$ is a general solution of the differential equation

$$x^2 \frac{dy}{dx} + 2xy = e^x$$

on the interval $x > 0$.