

# Problems and Explorations in Probability

A Problem-Centered Introduction

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*To Professor Kamta Rai,*

mentor, colleague and friend

you inspired me to study

probability theory.

# Preface

Probability has been part of my academic life for decades, and this book grew from that experience. The problems gathered here appeared in my classrooms, homework sets, exams, review sheets, and conversations prompted by questions that students asked after class. Many hundreds of students have worked through versions of these problems while studying probability with me at Western Kentucky University, Cal Poly Pomona, and elsewhere. In this sense, the text records a collection of probability problems that form a course which evolved through years of teaching and experience.

The text is deliberately problem-centered. Its five hundred problems form the mathematical core of the book, while the exposition and worked examples, figures, historical remarks, and short explorations are intended to give a reader enough structure to begin working and perspective to understand why a method matters. The problems move from relatively direct applications toward conceptual questions, proofs, and classical problems. Every problem has a corresponding answer, hint, or worked solution in the appendix so that the book can be used either in a classroom, or as a supplemental problem collection, or for independent study.

There is intentionally more material here than belongs in any one course, and there are several reasonable paths through it. A first course would cover Chapters 1–7. The 155 problems marked with a star,  $\star$ , provide a path through that material and make a reasonable foundation for a one-semester undergraduate probability course. It should be emphasized here that the star is not a difficulty rating; it identifies problems that collectively carry the main mathematical development. The surrounding unstarred problems can be used for additional practice, alternate examples, honors work, historical enrichment, or a second pass through the subject, while Chapter 8 offers optional extensions into Poisson processes, finite Markov chains, random walks, branching processes, martingales, and computational probability. An instructor can follow the starred route closely, substitute nearby problems to suit a class, or build a more specialized path through the larger collection.

Most of the manuscript can be studied after a two-semester calculus sequence with portions of Chapter 6 dealing with multivariate continuous distributions and transformations of two random variables, this requires familiarity with double integrals. Many of my students studied probability while currently enrolled in Calculus III.

The collection remains rooted in the classical traditions of probability. Dice, cards, marbles, roulette, waiting-time questions, gambling games, the birthday problem, the problem of points, the St. Petersburg paradox, Buffon's needle, and gambler's ruin appear because these examples have endured due to their mathematical richness. At the same time, the book follows some of those classical ideas into topics that are now part of a broader view of probability, including entropy and information, Pólya reinforcement and exchangeability, conditional expectation and total variance, order statistics, the bivariate normal distribution, concentration inequalities, martingales, branching processes, and Markov chains. The intention is not to turn a first course into a survey of advanced

probability, but to let students see where elementary ideas lead.

Four continuing examples provide an additional route through the text. A box of colored marbles, a Galton board and its associated random walk, an arrival process, and a fair game return are revisited, each time viewed with mathematical tools that were not available at their previous appearance. The marble box moves from sample spaces to counting, conditioning, distributions, laws of large numbers, and Pólya reinforcement; the Galton board becomes first a collection of paths, then a binomial model, a sum of small steps, a central-limit example, and finally a Markov chain; Poisson counts, exponential waiting times, gamma arrival times, competing clocks, and normal approximations are eventually recognized as different views of one arrival process; and a simple fair game develops from expectation into random walks and the martingale idea. These recurring models are meant to show a characteristic feature of probability theory: the experiment may remain the same while new mathematical ideas reveal additional aspects. In probability, it is all about the sample space.

Computation and simulation are used in the same spirit, numerical experiments can test a conjecture, using either a spreadsheet or statistical package, or generative AI tools such as ChatGPT [14] can help translate a probabilistic specification into code. Two extended examples detail the use of AI: Section 1.2.4 develops a marble-draw simulator, while Section 7.2.1 develops and checks a rectangular Galton-board simulator.

The hope is that the book retains the feeling of the courses from which it grew: a collection of useful and interesting probability problems. You do not need to solve all five hundred problems to learn probability. The key to understanding probability is to describe the random experiment carefully, decide what information is relevant, choose a useful representation, carry out the calculation, and then ask whether the answer makes probabilistic sense.

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# Notation Used

The notation below is used repeatedly throughout the manuscript. Additional notation is introduced locally when a topic requires it.

|                                                                                                                   |                                                                                                                                    |
|-------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| $S$                                                                                                               | $\text{Var}(X)$                                                                                                                    |
| The sample space, or set of all possible outcomes.                                                                | The variance of $X$ ; $\sigma^2$ is commonly used for the variance and $\sigma$ for the standard deviation.                        |
| $A, B, C, \dots$                                                                                                  | $\text{Cov}(X, Y), \rho$                                                                                                           |
| Events, represented as subsets of the sample space.                                                               | Covariance and correlation coefficient, respectively.                                                                              |
| $\emptyset$                                                                                                       | $\mathbf{1}_A$                                                                                                                     |
| The empty event.                                                                                                  | The indicator of event $A$ , equal to 1 when $A$ occurs and 0 otherwise.                                                           |
| $A \cup B, A \cap B$                                                                                              | $X \sim \mathcal{D}$                                                                                                               |
| Union (“ $A$ or $B$ ”) and intersection (“ $A$ and $B$ ”).                                                        | “ $X$ has distribution $\mathcal{D}$ .” Named distributions are written explicitly, for example $X \sim \text{Poisson}(\lambda)$ . |
| $\overline{A}$                                                                                                    | $q = 1 - p$                                                                                                                        |
| The complement of the event $A$ .                                                                                 | The failure probability in Bernoulli-trial models when $p$ denotes success probability.                                            |
| $P(A)$                                                                                                            | $\overline{X}$                                                                                                                     |
| The probability of the event $A$ .                                                                                | The sample mean, $n^{-1} \sum_{i=1}^n X_i$ .                                                                                       |
| $P(A   B)$                                                                                                        | $m_X(t)$                                                                                                                           |
| The conditional probability of $A$ given $B$ .                                                                    | A moment-generating function (mgf). The notation $m(t)$ is used when the random variable is clear.                                 |
| $N(A)$                                                                                                            | $G_X(s), \phi_X(s)$                                                                                                                |
| The number of outcomes in a finite event $A$ , when counting notation is being used.                              | Probability-generating functions (pgfs); both notations occur in the manuscript.                                                   |
| $X, Y, X_1, \dots$                                                                                                | $\Gamma(\alpha), B(\alpha, \beta)$                                                                                                 |
| Random variables; lowercase letters such as $x$ and $y$ denote particular values.                                 | The gamma and beta functions.                                                                                                      |
| $p(x)$                                                                                                            | $\Phi(z)$                                                                                                                          |
| A probability mass function for a discrete random variable. In some early discrete examples, $f(x)$ is also used. | The standard normal cumulative distribution function.                                                                              |
| $f(x), f(x, y)$                                                                                                   | $\xrightarrow{P}, \xrightarrow{d}$                                                                                                 |
| A probability density function, or a joint density in the two-variable case.                                      | Convergence in probability and convergence in distribution. Almost-sure convergence is written explicitly as “almost surely.”      |
| $F(x), F(x, y)$                                                                                                   | $\lfloor x \rfloor, \lceil x \rceil$                                                                                               |
| A cumulative distribution function, or a joint distribution function.                                             | The greatest integer not exceeding $x$ and the least integer not less than $x$ .                                                   |
| $E(X)$                                                                                                            |                                                                                                                                    |
| The expected value or mean of $X$ ; $\mu$ is commonly used for the mean.                                          |                                                                                                                                    |

# Suggested Core Course Path

This book contains more problems than would normally be assigned in a single undergraduate probability course. To make it easier to use as a self-study guide, a supplemental text, or the text for a semester course, a selected set of problems is marked with a star, ★, placed immediately before the word *Problem*. The starred problems form a coherent first-course path through the book. They include routine calculations, representative applications, several classical problems, and a modest number of proofs that develop ideas used later.

The star is *not* a difficulty rating. A starred problem may be challenging because it represents an idea that belongs in the mathematical development.. The unstarred problems provide additional practice, alternate contexts, historical examples, and extensions. Instructors can freely substitute nearby unstarred problems without disturbing the sequence.

The present selection contains **155 starred problems**. This collection of problems is one possible 15-week semester course:

| Weeks | Core emphasis                                                                                                                   | Chapters |
|-------|---------------------------------------------------------------------------------------------------------------------------------|----------|
| 1–2   | Sets, sample spaces, axioms, complements, inclusion–exclusion, and equally likely models                                        | 1        |
| 3     | Counting principles, permutations, combinations, and finite models                                                              | 2        |
| 4–5   | Conditional probability, independence, total probability, and Bayes’ theorem                                                    | 3        |
| 6–8   | Discrete random variables, expectation and variance; binomial, hypergeometric, geometric, negative binomial, and Poisson models | 4        |
| 9–10  | Continuous random variables; uniform, exponential, gamma, normal, and selected additional families                              | 5        |
| 11–12 | Joint and conditional distributions, covariance, sums and transformations, order statistics, and geometric probability          | 6        |
| 13–14 | Sample means, laws of large numbers, the central limit theorem, and normal approximation                                        | 7        |
| 15    | Selected Poisson-process, Markov-chain, random-walk, and simulation topics                                                      | 8        |

A shorter course can omit the starred Chapter 8 problems and the starred problems in optional explorations. Conversely, a second course or an honors section can use the surrounding unstarred problems to deepen the theoretical and computational development.

# Contents

|                                                                            |           |
|----------------------------------------------------------------------------|-----------|
| <b>Preface</b>                                                             | <b>ii</b> |
| <b>Notation Used</b>                                                       | <b>iv</b> |
| <b>Suggested Core Course Path</b>                                          | <b>v</b>  |
| <b>1 Sets, Sample Spaces, and Probability</b>                              | <b>1</b>  |
| 1.1 Sets and Events . . . . .                                              | 1         |
| 1.1.1 Exercises: Sets and Events . . . . .                                 | 3         |
| 1.2 Probability Models and Axioms . . . . .                                | 8         |
| 1.2.1 Experiments, Sample Spaces, and Events . . . . .                     | 8         |
| 1.2.2 The Probability Axioms . . . . .                                     | 9         |
| 1.2.3 Consequences of the Axioms . . . . .                                 | 9         |
| 1.2.4 Exploration: Building a Marble-Draw Simulator with ChatGPT . . . . . | 11        |
| 1.2.5 Exercises: Unions, Intersections, and Complements . . . . .          | 13        |
| 1.2.6 Exercises: Sample Spaces . . . . .                                   | 15        |
| 1.2.7 Exercises: Finite and Countable Probability Models . . . . .         | 19        |
| <b>2 Counting Methods and Finite Probability Models</b>                    | <b>22</b> |
| 2.1 Counting Principles . . . . .                                          | 22        |
| 2.1.1 The Fundamental Principle of Counting . . . . .                      | 23        |
| 2.1.2 Permutations . . . . .                                               | 23        |
| 2.1.3 Combinations . . . . .                                               | 23        |
| 2.1.4 The Multinomial Coefficient . . . . .                                | 23        |
| 2.1.5 Exercises: Counting and Finite Models . . . . .                      | 25        |
| <b>3 Conditional Probability, Independence, and Bayes' Theorem</b>         | <b>30</b> |

|          |                                                                |           |
|----------|----------------------------------------------------------------|-----------|
| 3.1      | Conditional Probability and Independence . . . . .             | 30        |
| 3.1.1    | The Multiplication Rule . . . . .                              | 31        |
| 3.1.2    | Exercises: Conditional Probability and Independence . . . . .  | 32        |
| 3.1.3    | Further Problems and Explorations . . . . .                    | 36        |
| 3.2      | Total Probability and Bayes' Theorem . . . . .                 | 38        |
| 3.2.1    | The Law of Total Probability . . . . .                         | 39        |
| 3.2.2    | Exploration: Base Rates, Odds, and Repeated Evidence . . . . . | 40        |
| 3.2.3    | Exercises: Total Probability and Bayes' Theorem . . . . .      | 40        |
| <b>4</b> | <b>Discrete Random Variables and Distributions</b>             | <b>45</b> |
| 4.1      | Random Variables, Distributions, and Expectation . . . . .     | 45        |
| 4.1.1    | Expected Value and Variance . . . . .                          | 46        |
| 4.1.2    | Properties of Expectation and Variance . . . . .               | 47        |
| 4.1.3    | Moments and Inequalities . . . . .                             | 48        |
| 4.1.4    | Exercises: Discrete Random Variables . . . . .                 | 49        |
| 4.2      | Standard Discrete Distributions . . . . .                      | 54        |
| 4.2.1    | The Bernoulli Distribution . . . . .                           | 55        |
| 4.2.2    | The Binomial Distribution . . . . .                            | 55        |
| 4.2.3    | The Multinomial Distribution . . . . .                         | 56        |
| 4.2.4    | The Hypergeometric Distribution . . . . .                      | 57        |
| 4.2.5    | The Geometric Distribution . . . . .                           | 57        |
| 4.2.6    | The Negative Binomial Distribution . . . . .                   | 58        |
| 4.2.7    | The Poisson Distribution . . . . .                             | 58        |
| 4.2.8    | The Poisson Approximation to the Binomial . . . . .            | 59        |
| 4.2.9    | Exploration: Pólya's Urn and Exchangeability . . . . .         | 60        |
| 4.2.10   | Exercises: Binomial Distribution . . . . .                     | 61        |
| 4.2.11   | Exercises: Multinomial Distribution . . . . .                  | 64        |
| 4.2.12   | Exercises: Hypergeometric Distribution . . . . .               | 64        |
| 4.2.13   | Exercises: Geometric Distribution . . . . .                    | 66        |
| 4.2.14   | Exercises: Negative Binomial Distribution . . . . .            | 70        |
| 4.2.15   | Exercises: Poisson Distribution . . . . .                      | 72        |
| 4.2.16   | Exercises: Poisson Approximation . . . . .                     | 73        |
| 4.2.17   | Further Problems on Discrete Distributions . . . . .           | 73        |
| 4.2.18   | Exploration: Surprise, Entropy, and Information . . . . .      | 76        |

|          |                                                        |            |
|----------|--------------------------------------------------------|------------|
| 4.3      | Generating Functions . . . . .                         | 77         |
| 4.3.1    | Moment-Generating Functions . . . . .                  | 77         |
| 4.3.2    | Probability-Generating Functions . . . . .             | 77         |
| 4.3.3    | Exercises: Moment-Generating Functions . . . . .       | 79         |
| 4.3.4    | Exercises: Probability-Generating Functions . . . . .  | 81         |
| 4.3.5    | Exploration: Relabeling Dice . . . . .                 | 82         |
| <b>5</b> | <b>Continuous Random Variables and Distributions</b>   | <b>88</b>  |
| 5.1      | Continuous Random Variables and Densities . . . . .    | 88         |
| 5.1.1    | Exercises: Continuous Random Variables . . . . .       | 91         |
| 5.2      | Standard Continuous Distributions . . . . .            | 93         |
| 5.2.1    | The Uniform Distribution . . . . .                     | 94         |
| 5.2.2    | The Exponential Distribution . . . . .                 | 94         |
| 5.2.3    | The Gamma Distribution . . . . .                       | 95         |
| 5.2.4    | The Normal Distribution . . . . .                      | 96         |
| 5.2.5    | Exercises: Uniform Distribution . . . . .              | 98         |
| 5.2.6    | Exercises: Exponential Distribution . . . . .          | 101        |
| 5.2.7    | Exercises: Gamma Distribution . . . . .                | 103        |
| 5.2.8    | Exercises: Normal Distribution . . . . .               | 104        |
| 5.2.9    | The Beta Distribution . . . . .                        | 107        |
| 5.2.10   | Exploration: Learning an Unknown Probability . . . . . | 107        |
| 5.2.11   | The Chi-Square Distribution . . . . .                  | 108        |
| 5.2.12   | Survival and Hazard Functions . . . . .                | 108        |
| 5.2.13   | Further Continuous Distributions . . . . .             | 109        |
| 5.2.14   | Exercises: Moment-Generating Functions . . . . .       | 113        |
| <b>6</b> | <b>Joint Distributions and Transformations</b>         | <b>115</b> |
| 6.1      | Joint and Marginal Distributions . . . . .             | 115        |
| 6.1.1    | Discrete Joint Distributions . . . . .                 | 115        |
| 6.1.2    | Continuous Joint Distributions . . . . .               | 116        |
| 6.1.3    | Independence . . . . .                                 | 117        |
| 6.1.4    | Covariance and Correlation . . . . .                   | 118        |
| 6.1.5    | The Bivariate Normal Distribution . . . . .            | 119        |
| 6.1.6    | Exercises: Joint Distributions . . . . .               | 121        |

|          |                                                                       |            |
|----------|-----------------------------------------------------------------------|------------|
| 6.2      | Conditional Distributions and Conditional Expectation . . . . .       | 128        |
| 6.2.1    | Exercises: Conditional Distributions and Expectation . . . . .        | 130        |
| 6.3      | Order Statistics . . . . .                                            | 131        |
| 6.3.1    | Exercises: Order Statistics . . . . .                                 | 132        |
| 6.4      | Geometric Probability . . . . .                                       | 133        |
| 6.4.1    | Exploration: Bertrand's Paradox and the Meaning of Random . . . . .   | 133        |
| 6.4.2    | Problems and Explorations . . . . .                                   | 134        |
| 6.5      | Functions and Sums of Random Variables . . . . .                      | 138        |
| 6.5.1    | Linear Combinations . . . . .                                         | 138        |
| 6.5.2    | The Method of Moment-Generating Functions . . . . .                   | 138        |
| 6.5.3    | The Sample Mean . . . . .                                             | 139        |
| 6.5.4    | Transformations of Two Random Variables and Jacobians . . . . .       | 140        |
| 6.5.5    | Exercises: Transformations and Jacobians . . . . .                    | 141        |
| 6.5.6    | Exercises: Functions and Sums . . . . .                               | 142        |
| <b>7</b> | <b>Sampling Distributions and Limit Theorems</b>                      | <b>146</b> |
| 7.1      | Sequences, Sample Means, and Laws of Large Numbers . . . . .          | 146        |
| 7.1.1    | Laws of Large Numbers . . . . .                                       | 146        |
| 7.1.2    | Modes of Convergence . . . . .                                        | 147        |
| 7.1.3    | The Strong Law of Large Numbers . . . . .                             | 148        |
| 7.1.4    | Exploration: From Jensen to Concentration Bounds . . . . .            | 149        |
| 7.1.5    | Exercises: Sampling Distributions and Laws of Large Numbers . . . . . | 150        |
| 7.1.6    | Exercises: Modes of Convergence and the Strong Law . . . . .          | 151        |
| 7.2      | Central Limit Theorem and Normal Approximation . . . . .              | 151        |
| 7.2.1    | Exploration: Building a Galton Board Simulator with ChatGPT . . . . . | 153        |
| 7.2.2    | Exercises: Central Limit Theorem and Normal Approximation . . . . .   | 157        |
| <b>8</b> | <b>Stochastic Processes and Computational Probability</b>             | <b>159</b> |
| 8.1      | Poisson Processes . . . . .                                           | 159        |
| 8.1.1    | Exercises: Poisson Processes . . . . .                                | 160        |
| 8.2      | Finite Markov Chains and Random Walks . . . . .                       | 161        |
| 8.2.1    | Exercises: Markov Chains and Random Walks . . . . .                   | 162        |
| 8.2.2    | Exploration: Fair Games and Martingales . . . . .                     | 163        |
| 8.3      | Branching Processes and Extinction . . . . .                          | 164        |

---

|          |                                                    |            |
|----------|----------------------------------------------------|------------|
| 8.4      | Computational Probability and Simulation . . . . . | 166        |
| 8.4.1    | Exercises: Simulation Principles . . . . .         | 166        |
| <b>A</b> | <b>Answers and Hints</b>                           | <b>168</b> |
|          | <b>References</b>                                  | <b>210</b> |
|          | <b>Subject Index</b>                               | <b>211</b> |

## Chapter 5

# Continuous Random Variables and Distributions

*For a continuous random variable, probability is represented by area under a density curve. This chapter develops density and distribution functions, expectation and variance, and the uniform, exponential, gamma, and normal families, along with several useful extensions.*

### Learning objectives.

- Verify and use probability density functions.
- Move between density and cumulative distribution functions.
- Compute expectations and variances by integration.
- Recognize and apply common continuous distributions.

## 5.1 Continuous Random Variables and Densities

For a continuous random variable, probability is represented by area rather than by the height of the density at a point. This distinction is fundamental: a density may exceed one, while a probability cannot, and every single point has probability zero even though intervals can have positive probability. The cumulative distribution function provides a second description of the same model and often gives the cleanest route to probabilities and transformations.

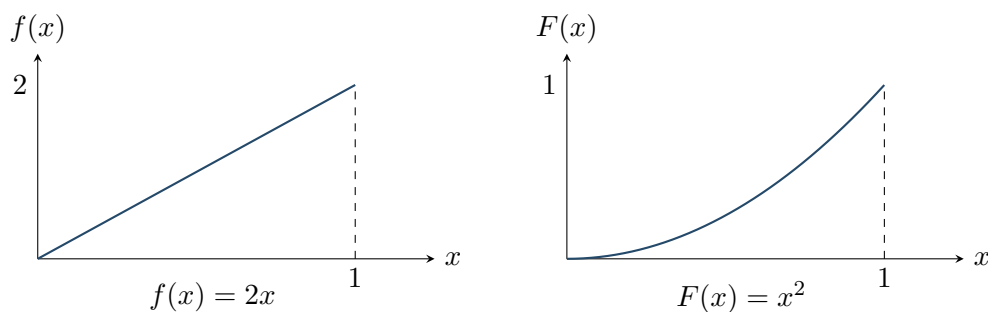


Figure 5.1: For the triangular density  $f(x) = 2x$  on  $(0, 1)$ , the cdf is the accumulated area  $F(x) = x^2$ .

A random variable  $X$  with distribution function  $F(x)$  is a continuous random variable if  $F(x)$  is continuous, for  $-\infty < x < \infty$

Let  $F(x)$  be the distribution function for a continuous random variable  $X$ . Then  $f(x)$ , defined by

$$f(x) = \frac{dF(x)}{dx} = F'(x) \quad (5.1.1)$$

when the derivative exists, is the probability density function (p.d.f.) for the random variable  $X$ .

**Properties of a Density Function:** If  $f(x)$  is a density function for a continuous random variable  $X$ , then

1.  $f(x) \geq 0$  for all  $-\infty < x < \infty$

2.  $\int_{-\infty}^{\infty} f(x)dx = 1$ .

For a random variable  $X$  with density function  $f(x)$  and  $a < b$  the probability that  $X$  is in the interval  $[a, b]$  is

$$P(a \leq X \leq b) = \int_a^b f(x)dx \quad (5.1.2)$$

Let  $f(x)$  be the probability density function of a continuous random variable  $X$ . Then the cumulative distribution function  $F(x)$  of  $X$  is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt \quad (5.1.3)$$

The expected value of a continuous random variable  $X$  is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad (5.1.4)$$

provided that the integral exists.

**The Law of the Unconscious Statistician:** Let  $g(X)$  be a function of  $X$ , the expected value of  $g(X)$  is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

provided that the integral exists.

For  $0 < p < 1$ , a  $100p^{th}$  percentile of the distribution of a random variable  $X$  is any number  $q$  so that

$$P(X \leq q) \leq p \text{ and } P(X > q) \leq 1 - p$$

We consider percentiles of a continuous random variable. If a random variable is discrete, then to discuss percentile, we need the *order statistics*, which we will not consider.

The 25th and 75th percentiles of any distribution are the *first* and the *third* quartiles.

The 50th percentile of any distribution is called the *median* of the distribution.

The median divides the distribution of the probability mass into two equal parts

A *mode* of the distribution of a continuous random variable  $X$  is the value of  $x$  where the probability density function  $f(x)$  attains a relative maximum.

A mode of a random variable  $X$  is one of its most probable values. A random variable can have more than one mode.

**Worked Example 5.1** (A triangular density). Let

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Because  $f(x) \geq 0$  and  $\int_0^1 2x \, dx = 1$ , this is a density. Moreover,

$$P(X > 0.7) = \int_{0.7}^1 2x \, dx = 1 - (0.7)^2 = 0.51,$$

and

$$E(X) = \int_0^1 2x^2 \, dx = \frac{2}{3}.$$

**Worked Example 5.2** (From a cdf to probabilities and a density). Suppose

$$F(x) = \begin{cases} 0, & x < 0, \\ x^2/4, & 0 \leq x \leq 2, \\ 1, & x > 2. \end{cases}$$

On  $(0, 2)$  the density is  $f(x) = F'(x) = x/2$ . Also,

$$P(0.5 < X < 1.5) = F(1.5) - F(0.5) = \frac{2.25 - 0.25}{4} = \frac{1}{2}.$$

For a continuous random variable, endpoint choices do not change this probability.

**Worked Example 5.3** (From density to cdf). Suppose

$$f(x) = 2x, \quad 0 < x < 1,$$

and  $f(x) = 0$  otherwise. Then for  $0 \leq x \leq 1$ ,

$$F(x) = P(X \leq x) = \int_0^x 2t \, dt = x^2.$$

Thus  $P(1/2 < X < 3/4) = F(3/4) - F(1/2) = 5/16$ . The density describes local concentration; the cdf accumulates that density into probabilities.

### 5.1.1 Exercises: Continuous Random Variables

★ **Problem 267.** Let  $X$  be the random variable with density function

$$f(x) = \begin{cases} (c+1)x^2 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- (a) What is the value of  $c$ ?
- (b) Find  $P(0.2 < X < 0.7)$
- (c) Find  $E(X)$
- (d) What is the median of  $X$ ?

**Problem 268.** For the continuous random variable  $X$  with probability density

$$f(x) = \begin{cases} x+1 & \text{for } -1 < x < 0 \\ 1-x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- (a) Find  $P(-1/2 < X \leq 3/4)$ .
- (b) Find the cumulative distribution function  $F(x)$ .
- (c) Find  $P(X \leq 2/3)$ .
- (d) The expected value  $E(X)$ .

**Problem 269.** Let  $X$  be the random variable with probability density function

$$f(x) = c(1-x^2) \text{ for } -1 < x < 1$$

- (a) What is the value of  $c$ ?
- (b) Find the cumulative distribution function of  $X$ .
- (c) Find  $P(-0.2 < X < 0.75)$ .
- (d) Find  $E(X)$ .

★ **Problem 270.** A random variable  $X$  has density

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find  $c$ .
- (b) Find  $P(|X| < 1/2)$ .
- (c) Find  $E(X)$  without integrating  $xf(x)$  directly.

**Problem 271.** Let the random variable  $X$  have the density function

$$f(x) = \begin{cases} cx & \text{for } 0 \leq x \leq \sqrt{\frac{2}{c}} \\ 0 & \text{elsewhere} \end{cases}$$

If the mode of this distribution is at  $x = \frac{\sqrt{2}}{4}$ , what is the median of  $X$ ?

**Problem 272.** Let  $X$  be a random variable with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

What is the  $P(0 < e^X < 4)$ ?

★ **Problem 273.** Suppose a random variable  $X$  has cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ x^2/4, & 0 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

- (a) Find the density of  $X$ .
- (b) Find the median of  $X$ .
- (c) Find  $E(X)$ .

**Problem 274.** The *logistic* distribution has cdf

$$F(x) = (1 + e^{-x})^{-1}, \text{ for } -\infty < x < \infty.$$

Find the pdf of the logistic distribution and show that its graph is symmetric about  $x = 0$

**Problem 275.** For what value of the constant  $c$  does the real valued function  $f(x)$  given by

$$f(x) = \frac{c}{1 + (x - \theta)^2} \text{ for } -\infty < x < \infty$$

where  $\theta$  is a real parameter, give a probability density function for random variable  $X$ ?

**Problem 276.** Given the probability density function of a random variable  $X$  as

$$f(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

what is the  $n$ th moment of  $X$  about the origin?

★ **Problem 277.** Let  $X$  be a random variable with probability density function  $f(x)$  and cumulative distribution function  $F(x)$ . True or False?

- (a)  $f(x)$  cannot be larger than 1.
- (b)  $F(x)$  cannot be larger than 1.
- (c)  $f(x)$  cannot decrease.
- (d)  $F(x)$  cannot decrease.

- (e)  $f(x)$  cannot be negative.
- (f)  $F(x)$  cannot be negative.
- (g) Area under  $f$  must be 1.
- (h) Area under  $F$  must be 1.
- (i)  $f$  cannot jump.
- (j)  $F$  cannot jump.

**Problem 278.** For a random variable  $X$  with a Cauchy distribution with  $\theta = 0$  (see Problem 275), so that

$$f(x) = \frac{\frac{1}{\pi}}{1 + x^2} \text{ for } -\infty < x < \infty$$

- (a) Show that the expected value of the random variable  $X$  does not exist.
- (b) Show that the variance of the random variable  $X$  does not exist.
- (c) Show that a Cauchy random variable does not have finite moments of order greater than or equal to one.

**Discussion:** Functions with the form of the density function of the Cauchy distribution were studied by mathematicians in the 17th century, but in a different context and under the title of the *witch of Agnesi*. Despite its name, the first explicit analysis of the properties of the Cauchy distribution was published by the French mathematician Poisson in 1824, with Cauchy only becoming associated with it during an academic controversy in 1853. Poisson noted that if the mean of observations following such a distribution were taken, the mean error did not converge to any finite number. As such, Laplace's use of the Central Limit Theorem with such a distribution was inappropriate, as it assumed a finite mean and variance. The importance of assuming a distribution has at least a finite second moment cannot be over emphasized.

**Problem 279.** Let  $f(x)$  be a continuous probability density function. Show that, for every  $-1 < \mu < 1$  and  $\sigma^2 > 0$ , the function  $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$  is also a probability density function.

★ **Problem 280.** Let  $X$  have a continuous, strictly increasing cumulative distribution function  $F$ . Define  $U = F(X)$ . Prove that  $U$  is uniformly distributed on  $[0, 1]$ . This result is called the *probability integral transform*.

## 5.2 Standard Continuous Distributions

As with discrete models, the continuous families are more memorable when tied to a mechanism. Uniform models express equal density over an interval, exponential models describe memoryless waiting, gamma models accumulate several waiting times, and the normal family describes a symmetric concentration about a mean. Later families such as beta, chi-square, Weibull, lognormal, and Pareto broaden the shapes available for proportions, lifetimes, and skewed measurements.

### 5.2.1 The Uniform Distribution

For  $\alpha < \beta$ , a random variable  $X$  has a continuous uniform probability distribution on the interval  $(\alpha, \beta)$  if the density function of  $X$  is

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha < x < \beta, \\ 0, & \text{elsewhere.} \end{cases} \quad (5.2.1)$$

The uniform distribution is used to model the probability for selecting points at random from an interval  $[\alpha, \beta]$ .

The uniform distribution is used to model the probability for selecting points at random from an interval  $[\alpha, \beta]$ .

The expected value of a uniform random variable  $X$  is

$$E[X] = \frac{\alpha + \beta}{2}$$

the variance is

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

The  $p$ th percentile of a uniform random variable is  $p(\beta - \alpha) + \alpha$  so that the median is  $(\alpha + \beta)/2$

The moment-generating function is

$$m(t) = \frac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$$

### 5.2.2 The Exponential Distribution

A random variable  $X$  has an exponential distribution with parameter  $\theta > 0$  if  $X$  has a probability density function

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases} \quad (5.2.2)$$

The expected value of an exponential random variable  $X$  is

$$E[X] = \theta$$

the variance is

$$\text{Var}(X) = \theta^2$$

The moment-generating function is

$$m(t) = \frac{1}{1 - \theta t}$$

The  $p$ th percentile of an exponential random variable is  $-\theta \ln(1 - p)$  so that the median is  $\theta \ln 2$

The exponential distribution is often used to model the waiting time  $W$  to the first arrival in a Poisson process with rate  $\lambda$ . In that setting,  $W$  has density

$$f(w) = \begin{cases} \lambda e^{-\lambda w} & \text{for } w > 0 \\ 0 & \text{elsewhere} \end{cases}$$

A continuous random variable  $X$  has an exponential distribution if and only if  $X$  is *memoryless*. That is, if and only if for any positive real numbers  $a$  and  $b$ ,

$$P(X > a + b \mid X > a) = P(X > b). \quad (5.2.3)$$

### 5.2.3 The Gamma Distribution

A random variable  $X$  has a gamma distribution with parameters  $\alpha > 0$  and  $\beta > 0$  if  $X$  has a probability density function

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases} \quad (5.2.4)$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

is the gamma function.

In Problem 313 we show that  $\Gamma(1) = 1$  and if  $\alpha > 1$ , then

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1),$$

so if  $n$  is a positive integer, then

$$\Gamma(n) = (n - 1)!.$$

When  $\alpha = 1$ , the gamma distribution reduces to the exponential distribution with  $\theta = \beta$

The expected value of a gamma random variable  $X$  is

$$E[X] = \alpha\beta$$

the variance is

$$\text{Var}(X) = \alpha\beta^2$$

The moment-generating function is

$$m(t) = \frac{1}{(1 - \beta t)^\alpha}$$

When  $\alpha = k$  is a positive integer, the gamma distribution models the waiting time  $W$  to the  $k$ th arrival in a Poisson process with rate  $\lambda$ . In that setting,  $W$  has density

$$f(w) = \begin{cases} \frac{\lambda^k}{(k-1)!} w^{k-1} e^{-\lambda w} & \text{for } w > 0 \\ 0 & \text{elsewhere} \end{cases}$$

which is often called the *Erlang* distribution.

### 5.2.4 The Normal Distribution

A random variable  $X$  has a normal distribution if the density function of  $X$  is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty. \quad (5.2.5)$$

where  $\sigma > 0$  and  $-\infty < \mu < \infty$ .

The most widely used continuous probability distribution is the normal distribution, a distribution with the familiar bell shape.

The expected value of a normal random variable  $X$  is

$$E[X] = \mu$$

the variance is

$$\text{Var}(X) = \sigma^2$$

The moment-generating function is

$$m(t) = e^{\mu t + \sigma^2 t^2 / 2}$$

The normal distribution is symmetric about  $\mu$ , the mean, with both the median and the mode coinciding with values  $\mu$ .

**Empirical Rule** For a distribution of measurements that is approximately normal (bell shaped), it follows that the interval with end points

- (a)  $\mu \pm \sigma$  contains approximately 68% of the measurements
- (b)  $\mu \pm 2\sigma$  contains approximately 95% of the measurements
- (c)  $\mu \pm 3\sigma$  contains almost all ( $\geq 99\%$ ) of the measurements

A normal distribution with mean zero,  $\mu = 0$  and standard deviation  $\sigma = 1$  is called a *standard* normal distribution and is often denoted by  $Z$ .

Standard normal probabilities are tabulated in Figure 5.2. The entries give  $A(z) = P(0 < Z < z) = \Phi(z) - \frac{1}{2}$  for positive  $z$ . Thus, for  $z > 0$ ,  $\Phi(z) = \frac{1}{2} + A(z)$ ; probabilities for negative  $z$ -scores follow from symmetry.

**Worked Example 5.4** (A uniform waiting location). Let  $X$  be uniform on  $(2, 8)$ . Since probability is proportional to interval length,

$$P(3 < X < 5) = \frac{5 - 3}{8 - 2} = \frac{1}{3}.$$

The mean is the midpoint,

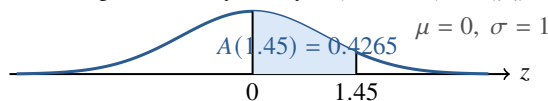
$$E(X) = \frac{2 + 8}{2} = 5.$$

## Standard Normal Table

Entries give the area between the mean and a positive  $z$ -score:

$$A(z) = P(0 < Z < z) = \Phi(z) - \frac{1}{2}, \quad Z \sim N(0, 1).$$

For negative  $z$ , use symmetry:  $P(z < Z < 0) = A(|z|)$ .



| $z$ | 0.00   | 0.01   | 0.02   | 0.03   | 0.04   | 0.05   | 0.06   | 0.07   | 0.08   | 0.09   |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0753 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2517 | 0.2549 |
| 0.7 | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2995 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3980 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4429 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4484 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4699 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4756 | 0.4761 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4864 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.4916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 0.4978 | 0.4979 | 0.4979 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4982 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986 |
| 3.0 | 0.4987 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |
| 3.1 | 0.4990 | 0.4991 | 0.4991 | 0.4991 | 0.4992 | 0.4992 | 0.4992 | 0.4992 | 0.4993 | 0.4993 |
| 3.2 | 0.4993 | 0.4993 | 0.4994 | 0.4994 | 0.4994 | 0.4994 | 0.4994 | 0.4995 | 0.4995 | 0.4995 |
| 3.3 | 0.4995 | 0.4995 | 0.4995 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4997 |
| 3.4 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4998 |
| 3.5 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 |
| 3.6 | 0.4998 | 0.4998 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.7 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.8 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.9 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |

Example: for  $z = 1.45$ , use row 1.4 and column 0.05 to obtain  $P(0 < Z < 1.45) = 0.4265$ .

Figure 5.2: Standard normal table. Entries give the area between the mean and a positive  $z$ -score.

Uniform models are appropriate when all locations in an interval are treated as equally likely.

**Worked Example 5.5** (Exponential and gamma waiting times). Suppose events occur according to a Poisson process at rate two per hour. The waiting time  $T_1$  to the first event is exponential with mean  $1/2$  hour. The waiting time  $T_3$  to the third event is gamma with shape  $\alpha = 3$  and scale  $\beta = 1/2$ , so

$$E(T_3) = \alpha\beta = \frac{3}{2} \text{ hours}, \quad \text{Var}(T_3) = \alpha\beta^2 = \frac{3}{4} \text{ hour}^2.$$

The exponential model describes one waiting stage; the gamma model describes the accumulated wait through several arrivals.

**Worked Example 5.6** (Standardizing a normal measurement). Suppose  $X \sim N(100, 15^2)$ . A value of 130 has standardized score

$$z = \frac{130 - 100}{15} = 2.$$

Thus 130 lies two standard deviations above the mean. The empirical rule also gives

$$P(85 < X < 115) \approx 0.68,$$

because the interval is  $\mu \pm \sigma$ .

**Worked Example 5.7** (Counts and waiting times suggest different models). Suppose arrivals occur at an average rate of three per hour under a Poisson-process model. The waiting time to the first arrival is exponential with rate 3, while the waiting time to the fourth arrival is gamma with shape 4 and rate 3. The rate parameter is the same, but the question determines the distribution. This connection between counts and waiting times will be developed explicitly in Chapter 8.

**Continuing Example: Arrivals: From Counts to Waiting Times.** The Poisson count model from Section 4.2.8 can be viewed from the clock instead of from the counter. If arrivals occur at rate  $\lambda$ , then

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

so the wait  $T_1$  to the first arrival is exponential. The waiting time  $T_k$  to the  $k$ th arrival is gamma (Erlang) with shape  $k$  and rate  $\lambda$ . Thus the Poisson, exponential, and gamma distributions are not isolated formulas: they answer different questions about the same arrival mechanism.

### 5.2.5 Exercises: Uniform Distribution

★ **Problem 281.** Suppose that  $X$  has a uniform distribution over the interval  $(0, 1)$ .

- (a) Find the cdf  $F(x)$ .
- (b) Show that  $P(a \leq X \leq a + b)$ , for  $a \geq 0, b \geq 0$ , and  $a + b \leq 1$  depends only upon the value of  $b$ .

**Problem 282.** Arrivals of customers at a checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter. Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

**Problem 283.** If a parachutist lands at a random point on a line between markers  $A$  and  $B$ ,

- (a) Find the probability that they land closer to  $A$  than to  $B$ .
- (b) Find the probability that their distance to  $A$  is more than three times the distance to  $B$ .
- (c) Suppose that three parachutists operate independently, what is the probability that exactly one of the three lands past the midpoint between  $A$  and  $B$ ?

★ **Problem 284.** Cars arrive randomly at a stoplight which is red for 200 seconds. Let  $X$  be the amount of time a randomly selected car has to wait at the light before it turns green. Suppose  $X$  has a uniform distribution on  $[0, 200]$

- (a) Find the probability that the car must wait longer than 2 minutes.
- (b) What is the probability that the car has to wait between 1 and 2 minutes at the light?

**Problem 285.** If  $X$  is uniform on  $(\alpha, \beta)$ , show the  $p^{th}$  percentile of  $X$  is  $p(\beta - \alpha) + \alpha$  and thus the median is  $(\alpha + \beta)/2$

**Problem 286.** If  $X$  has a uniform distribution on  $[0, 10]$ , what is  $P(X + \frac{10}{X} \geq 7)$ ?

★ **Problem 287.** The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time,  $X$ , is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost  $c_0$  of a new board and a cost that increases proportionally to  $X^2$ . If  $C$  is the cost incurred then

$$C = c_0 + c_1 X^2$$

- (a) Find the probability that the delivery time exceeds two days.
- (b) In terms of  $c_0$  and  $c_1$ , find the expected cost associated with a single failed circuit board.

**Problem 288.** If a randomly drawn circle has a radius that is uniformly distributed over  $(0, 1)$

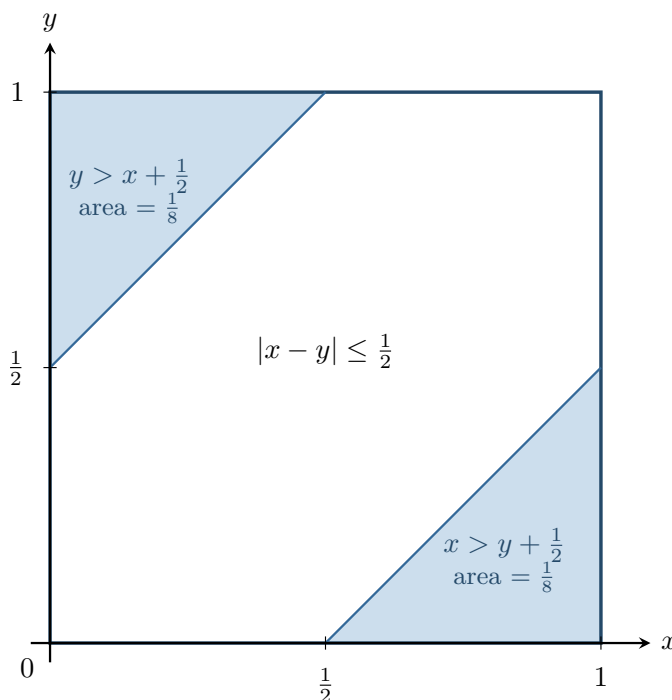
- (a) What is the expected area of the circle?
- (b) What is the variance of the area of the circle?

**Problem 289.** When a pair of coils is placed around a homing pigeon and a magnetic field applied that reverses the earth's field, it is thought that the bird will become disoriented. Under these circumstances the bird is just as likely to fly in one direction as in any other. Let  $\theta$  denote the direction in radians of the bird's initial flight.

- (a) What is the distribution of  $\theta$ ?
- (b) Find the probability that a bird will orient within  $\pi/4$  radians of home.

**Problem 290.** A box is to be constructed so that the height is 10 inches and its base is  $X$  inches by  $X$  inches. If  $X$  has a uniform distribution over the interval  $[2, 8]$ , then what is the expected volume of the box in cubic inches?

★ **Problem 291.** Two numbers are chosen independently and at random from the interval  $(0, 1)$ . What is the probability that the two numbers differ by more than  $1/2$ ?



Geometric representation of the event  $|X - Y| > \frac{1}{2}$  in the unit square.

**Problem 292.** If the random variable  $X$  has uniform distribution on the interval  $[0, a]$ , what is the probability that the random variable is greater than its square; that is, find

$$P(X > X^2)$$

**Problem 293.** If  $X$  has a uniform distribution on  $(0, 3)$ , find the probability that the roots of the equation  $g(t) = 0$  are real, where

$$g(t) = 4t^2 + 4Xt + X + 2.$$

★ **Problem 294.** If  $X$  is uniform on the interval on  $(-5, 5)$ , what is the probability that the quadratic equation  $100t^2 + 20tX + 2X + 3 = 0$  has complex solutions?

**Worked Example 5.8 (Waiting for an arrival).** For an exponential waiting time with mean  $\theta$ , the survival probability is

$$P(X > t) = e^{-t/\theta}.$$

Consequently, an average waiting time of ten minutes gives  $P(X > 5) = e^{-1/2}$ .

### 5.2.6 Exercises: Exponential Distribution

★ **Problem 295.** Let  $X$  have an exponential distribution with a mean  $\theta$  of 20. Find probability that  $X$  is less than 18.

**Problem 296.** The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will

- (a) exceed 3.0 on the Richter scale.
- (b) fall between 2.0 and 3.0 on the Richter scale.

**Problem 297.** The number of automobiles crossing a certain intersection during any time interval of length  $t$  minutes between 3:00 P.M. and 4:00 P.M. has a Poisson distribution with mean  $t$ . Let  $W$  be time elapsed after 3:00 P.M. before the first automobile crosses the intersection. What is the probability that  $W$  is less than 2 minutes?

★ **Problem 298.** If  $X$  has an exponential distribution and  $P(X > 2) = 0.0821$ , find

- (a) the value of  $\theta$
- (b)  $P(X < 1.7)$

**Problem 299.** The operator of a pumping station has observed that demand for water during early afternoon hours has an approximately exponential distribution with mean 100 cfs (cubic feet per second).

- (a) Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.
- (b) What water pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is only 0.01?

**Problem 300.** Show that the  $p$ th percentile of an exponential random variable  $X$  with parameter  $\theta > 0$  is  $-\theta \ln(1 - p)$  and that the median is  $\theta \ln 2$

★ **Problem 301.** Let  $X$  have an exponential distribution with mean  $\theta > 0$ . Show that  $X$  satisfies the memoryless property

$$P(X > x + y | X > x) = P(X > y).$$

**Problem 302.** Let  $F(x)$  be the cdf of the continuous-type random variable  $X$ , and assume that  $F(x) = 0$  for  $x \leq 0$  and  $0 < F(x) < 1$  for  $x > 0$ . Show that if  $X$  is memoryless, so that

$$P(X > x + y | X > x) = P(X > y)$$

then

$$F(x) = 1 - e^{-\lambda x}, \text{ for } x > 0.$$

Comment: Recall from Problem 196 that the geometric distribution, a discrete distribution, has this memoryless property. Problems 301 and 302 show that the exponential distribution is the only continuous distribution with the memoryless property. To show this, show that

$$g(x) = 1 - F(x)$$

satisfies Cauchy's functional equation

$$g(x + y) = g(x)g(y)$$

whose continuous positive solutions have the form

$$g(x) = e^{-\lambda x}.$$

**Problem 303.** Suppose that miracles do happen, but that there is no telling when they will happen because the time of the next occurrence of a miracle is independent of the last occurrence of a miracle. If the probability of a miracle within the next 1,000 years is 0.99, what is the probability that there will be a miracle within the next 200 years?

★ **Problem 304.** Show that for all positive values of  $t$  and small values of  $\Delta t$  that

$$P(T \leq t + \Delta t | T > t) \approx \frac{\Delta t}{\theta}$$

for an exponential random variable  $T$  with parameter  $\theta$ .

**Problem 305.** Let  $X$  be an exponentially distributed random variable with mean  $\theta$ . Define a random variable  $Y$  by

$$Y = k \text{ if } k - 1 \leq X < k \text{ for } k = 1, 2, \dots$$

- (a) Find  $P(Y = k)$  for each  $k = 1, 2, \dots$
- (b) Show that the expression in part a) can be written as

$$P(Y = k) = \left(e^{-1/\theta}\right)^{k-1} \left(1 - e^{-1/\theta}\right), k = 1, 2, \dots$$

so that  $Y$  has a geometric distribution with  $p = 1 - e^{-1/\theta}$

**Problem 306.** There are times when a shifted exponential model is appropriate. That is, let the pdf of  $X$  be

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-(x-a)/\theta} & \text{for } x > a \\ 0 & \text{elsewhere} \end{cases}$$

Calculate the mean and variance of  $X$ .

★ **Problem 307.** Let  $X$  and  $Y$  be independent exponential random variables with rates  $\lambda$  and  $\mu$ , respectively.

- (a) Find  $P(X < Y)$ .
- (b) Find the distribution of  $\min(X, Y)$ .

### 5.2.7 Exercises: Gamma Distribution

★ **Problem 308.** The response times for an interactive online game have approximately a gamma distribution with mean four seconds and variance eight seconds<sup>2</sup>.

- (a) Find the probability density function for the response times.
- (b) What is the probability that the response time is less than five seconds?

**Problem 309.** Cars arrive at a toll booth at a mean rate of three cars every 4 minutes according to a Poisson distribution

- (a) What is the probability that there are fewer than two cars in a 4 minute period?
- (b) Find the probability that the toll booth collector has to wait longer than 10 minutes to collect the 9th toll.

**Problem 310.** The weekly amount of downtime  $X$  (in hours) for an industrial machine has approximately a gamma distribution with  $\alpha = 3$  and  $\beta = 2$ . The loss  $L$  (in dollars) to the industrial operation as a result of this downtime is given by

$$L = 30X + 2X^2$$

Find the expected value and variance of  $L$ .

★ **Problem 311.** Annual incomes for heads of household in a section of a city have approximately a gamma distribution with  $\alpha = 20$  and  $\beta = 1000$ . Find the mean and variance of these incomes.

**Problem 312. The  $\chi^2$ -distribution** Let  $\nu$  be a positive integer. A random variable  $X$  is said to have a chi-square distribution with  $\nu$  degrees of freedom if  $X$  is a gamma-distributed random variable with parameters  $\alpha = \nu/2$  and  $\beta = 2$ . Tables that give probabilities associated with  $\chi^2$ -distributions are readily available in most statistics texts. Tables of the general gamma distribution are not so readily available.

- (a) If  $X$  has a gamma distribution with  $\alpha = n/2$  for some integer  $n$ , show  $2X/\beta$  has a  $\chi^2$ -distribution with  $n$  degrees of freedom.
- (b) Suppose  $X$  has a gamma distribution with  $\alpha = 1.5 = 3/2$  and  $\beta = 4$ , then  $2X/\beta = X/2$  has a  $\chi^2$ -distribution with 3 degrees of freedom. Compute  $P(X < 3.5) = P([X/2] < 1.75)$

★ **Problem 313.** The gamma function  $\Gamma(\alpha)$  for  $\alpha > 0$  is defined as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

- (a) Show that  $\Gamma(1) = 1$

(b) If  $\alpha > 1$ , show that

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1),$$

so if  $n$  is a positive integer, then

$$\Gamma(n) = (n - 1)!.$$

**Problem 314.** Show

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Comment: Make the substitution  $x = u^2$  in the integral representation for the gamma function to show that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} du$$

so that

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv$$

Now convert this double integral into polar coordinates so that

$$\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta$$

evaluate this last integral to obtain the solution.

**Problem 315.** If  $\lambda > 0$  and  $\alpha$  is a positive integer, show that

$$\frac{1}{\Gamma(\alpha)} \int_\lambda^\infty x^{\alpha-1} e^{-x} dx = \sum_{k=0}^{\alpha-1} \frac{\lambda^k e^{-\lambda}}{k!}$$

which gives a useful computation connection between incomplete gamma integrals and sums of Poisson probabilities.

★ **Problem 316.** Suppose that  $X$  has a gamma distribution with parameters  $\alpha > 0$  and  $\beta > 0$ . Show that if  $a$  is any value so that  $\alpha + a > 0$  then

$$E[X^a] = \frac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}$$

Compare this to Problem 276

### 5.2.8 Exercises: Normal Distribution

★ **Problem 317.** A certain training institute gives a standardized test to large numbers of applicants nationwide. The resulting scores form a normal distribution with mean 75 and standard deviation 10. Find the percent of all applicants with scores as follows.

- (a) Between 55 and 95.
- (b) Greater than 95.
- (c) Between 85 and 95.
- (d) Less than 75.

**Problem 318.** The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation 0.8.

- (a) What fraction of the students will possess a GPA in excess of 3.0?
- (b) If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?
- (c) Suppose that three students are randomly selected from the student body. What is the probability that all three will possess a GPA in excess of 3.0?

**Problem 319.** A company that manufactures and bottles juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce. Determine the proportion of bottles that will have more than 17 ounces dispensed into them.

★ **Problem 320.** The “fill” problem is important in many industries, such as those making cereal, cat litter, beverages, and so on. If an industry claims that it is selling 12 ounces of its product in a container, it must have a mean greater than 12 ounces, or else the Food and Drug Administration (FDA) will crack down, although the FDA will allow a very small percentage of the containers to have less than 12 ounces. If the content  $X$  of a container has a normal distribution with mean 12.1 find  $\sigma$  so that

$$P(X < 12) = 0.01.$$

**Problem 321.** In the design of a new aircraft carrier, the Navy needs to determine the length of bunks for berthing the crew. It is known that the male population is approximately normally distributed with a mean height of 74 inches and a standard deviation of 2 inches. How long should a bunk be in order to accommodate at least 90% of all naval personnel?

★ **Problem 322.** Among all applicants to a certain college one year, the Math SAT scores averaged 535 with a standard deviation of 100. The scores follow a normal distribution.

- (a) What percent of applicants have a Math SAT score above 600?
- (b) If an applicant has a Math SAT score of 650, in what percentile is this applicant?
- (c) The college plans to only admit applicants that are in the 95th percentile of Math SAT scores, what score does an applicant need to be admitted?

**Problem 323.** Show that the normal density function integrates to one.

**Discussion:** Recognizing that the solid  $S$  can be generated as a solid of revolution makes the calculation of the probability integral a straightforward and elementary integration. Are there other continuous functions  $f(x)$  for which the integral

$$\int_{-\infty}^{\infty} f(x) dx$$

can be computed in this manner?

Suppose that  $f(x)$  is a nonnegative continuous function for which the improper integral

$$I = \int_{-\infty}^{\infty} f(x) dx.$$

exists. Now, following the analysis applied to the probability integral, we have by Fubini's theorem

$$I^2 = \left( \int_{-\infty}^{\infty} f(x) dx \right) \left( \int_{-\infty}^{\infty} f(y) dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y) dx dy.$$

The solid  $S$  is described by

$$\{(x, y, z) | 0 \leq z \leq f(x)f(y), \text{ for } -\infty < x < \infty, -\infty < y < \infty\},$$

and will have a volume that can be computed using the method of cylindrical shells if the function  $z = f(x)$  is radially symmetric about the  $z$ -axis, for all  $x \geq 0$ .

Specifically, for the value of the double integral to be computed as

$$\int_0^{\infty} 2\pi x f(x) dx$$

the nonnegative continuous function  $f(x)$  must satisfy the functional equation

$$f(\sqrt{x^2 + y^2}) = f(x)f(y),$$

for all  $-\infty < x, y < \infty$ .

One way to solve this functional equation is to let  $g(x) = f(\sqrt{x})$  so that

$$\begin{aligned} g(x)g(y) &= f(\sqrt{x})f(\sqrt{y}) = f(\sqrt{(\sqrt{x})^2 + (\sqrt{y})^2}) \\ &= f(\sqrt{x+y}) = g(x+y). \end{aligned}$$

We have encountered the functional equation

$$g(x+y) = g(x)g(y)$$

previously (see Problem 302)  $g(x)$  must be of the form

$$g(x) = e^{ax},$$

where  $a$  is some constant. Thus, for the nonnegative continuous function  $f(x)$  that satisfies the functional equation, it follows that

$$f(x) = g(x^2) = e^{ax^2}.$$

From this, for the improper integral  $I$  to exist, the function  $f(x)$  must have  $a < 0$ . So, the only nonnegative continuous functions  $f(x)$  for which the improper integral

$$\int_{-\infty}^{\infty} f(x) dx$$

can be computed as a double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y) dx dy$$

are of the form

$$f(x) = e^{ax^2}$$

for  $-\infty < x < \infty$ , with  $a < 0$ . The continuous functions  $f(x)$  for which the “standard” calculation method used in the probability integral works *because* of the form of the integrand.

**Problem 324.** If  $X$  has a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ , show that the normal density obtains its maximum at  $x = \mu$  and has inflection points at  $x = \mu \pm \sigma$ .

★ **Problem 325.** Let  $X \sim N(\mu, \sigma^2)$ .

(a) Find  $a$  so that  $P(\mu - a\sigma < X < \mu + a\sigma) = 0.90$ .

(b) Find  $b$  so that  $P(X > \mu + b\sigma) = 0.01$ .

Give answers to two decimal places.

### 5.2.9 The Beta Distribution

For  $\alpha > 0$  and  $\beta > 0$ , the beta distribution has density

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad (5.2.6)$$

where  $B(\alpha, \beta)$  is the beta function. It is a flexible family for proportions and probabilities. Its mean and variance are

$$E(X) = \frac{\alpha}{\alpha + \beta}, \quad \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

The uniform distribution on  $(0, 1)$  is the special case  $\alpha = \beta = 1$ . The beta family will reappear in Chapter 6 through order statistics and in optional Bayesian explorations.

### 5.2.10 Exploration: Learning an Unknown Probability

The beta distribution is especially useful when an unknown probability is itself treated as uncertain. Suppose  $P$ , the chance of success on a Bernoulli trial, has a beta distribution with parameters  $\alpha, \beta$ . Conditional on  $P = p$ , observe  $X$  successes in  $n$  independent trials, so

$$P(X = x \mid P = p) \propto p^x(1-p)^{n-x}.$$

Multiplying this likelihood by the beta density shows that, after observing  $x$  successes and  $n - x$  failures,

$$P \mid X = x \sim \text{Beta}(\alpha + x, \beta + n - x). \quad (5.2.7)$$

This is a continuous version of the same “prior plus evidence gives posterior” idea introduced with Bayes’ theorem in Chapter 3. The parameters can be read as prior success and failure counts, and the predictive probability of success on the next trial is the posterior mean

$$\frac{\alpha + x}{\alpha + \beta + n}.$$

The calculation is elementary, but it is a useful bridge from classical probability to contemporary Bayesian modeling; see DeGroot and Schervish [3].

**Worked Example 5.9** (A uniform prior and ten observations). Start with  $P \sim \text{Beta}(1, 1)$ , the uniform distribution on  $(0, 1)$ . If seven successes and three failures are observed, then

$$P \mid X = 7 \sim \text{Beta}(8, 4),$$

and the predictive probability of success on the next trial is  $8/(8+4) = 2/3$ . The observed proportion is 0.7; the posterior predictive probability is pulled slightly toward the prior center  $1/2$ .

### Problems for the exploration

**Problem 326.** Suppose  $P \sim \text{Beta}(1, 1)$ . Conditional on  $P = p$ , a coin is tossed 12 times and 9 heads are observed. Find the posterior distribution of  $P$ , its posterior mean, and the predictive probability that the next toss is heads.

**Problem 327.** Suppose  $P \sim \text{Beta}(2, 2)$ . Five Bernoulli trials produce four successes and one failure. Find the posterior density up to its normalizing constant, identify the beta distribution, and compare the prior mean, observed proportion, and posterior mean.

**Problem 328.** Let  $P \sim \text{Beta}(\alpha, \beta)$ , and conditional on  $P = p$ , let  $X \sim \text{Binomial}(n, p)$ . Show that the unconditional distribution of  $X$  is

$$P(X = k) = \binom{n}{k} \frac{B(\alpha + k, \beta + n - k)}{B(\alpha, \beta)}, \quad k = 0, \dots, n.$$

This is the beta-binomial distribution.

### 5.2.11 The Chi-Square Distribution

A chi-square random variable with  $\nu > 0$  degrees of freedom is a gamma random variable with shape  $\nu/2$  and scale 2:

$$X \sim \chi_\nu^2 \iff X \sim \text{Gamma}\left(\frac{\nu}{2}, 2\right). \quad (5.2.8)$$

Thus  $E(X) = \nu$  and  $\text{Var}(X) = 2\nu$ . If  $Z_1, \dots, Z_\nu$  are independent standard normal random variables, then  $Z_1^2 + \dots + Z_\nu^2$  has a chi-square distribution. This connection is fundamental in statistical inference.

### 5.2.12 Survival and Hazard Functions

For a nonnegative continuous lifetime  $T$  with cdf  $F$  and density  $f$ , the survival function is

$$S(t) = P(T > t) = 1 - F(t), \quad (5.2.9)$$

and, where  $S(t) > 0$ , the hazard rate is

$$h(t) = \frac{f(t)}{S(t)}. \quad (5.2.10)$$

The hazard describes the instantaneous rate of failure conditional on survival to time  $t$ . For an exponential lifetime with rate  $\lambda$ ,  $S(t) = e^{-\lambda t}$  and  $h(t) = \lambda$  is constant. This constant hazard is another expression of the exponential distribution's memoryless property.

**Worked Example 5.10 (A Weibull hazard).** For a Weibull density  $f(t) = k\lambda t^{k-1}e^{-\lambda t^k}$ ,  $t > 0$ , the survival function is  $S(t) = e^{-\lambda t^k}$  and hence

$$h(t) = k\lambda t^{k-1}.$$

The hazard increases with time when  $k > 1$ , is constant when  $k = 1$ , and decreases when  $0 < k < 1$ .

### 5.2.13 Further Continuous Distributions

**Problem 329.** A random variable  $X$  has a *beta* distribution if its probability density function is given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \text{ for } 0 < x < 1$$

where  $\alpha > 0$  and  $\beta > 0$  and

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

is the beta function.

(a) Letting  $x = \sin^2 \theta$ , show that

$$B(\alpha, \beta) = 2 \int_0^{\frac{\pi}{2}} \cos^{2\alpha-1} \theta \sin^{2\beta-1} \theta d\theta$$

(b) Write  $\Gamma(\alpha)\Gamma(\beta)$  as a double integral and transform to polar coordinates to show that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

**Problem 330.** If  $X$  is a beta-distributed random variable with parameters  $\alpha > 0$  and  $\beta > 0$ ,

(a) Show the expected value is

$$\mu = \frac{\alpha}{\alpha + \beta}$$

(b) Show the variance is

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

**Problem 331.** The proportion of time per day that all checkout counters in a supermarket are busy follows a distribution

$$f(x) = kx^2(1-x)^9 \text{ for } 0 < x < 1$$

Find  $k$  so that this is a valid probability density function.

**Problem 332.** The percentage of impurities per batch in a certain chemical product is a random variable  $X$  that follows the beta distribution with  $\alpha = 4$  and  $\beta = 3$ . What is the probability that a randomly selected batch will have more than 25% impurities?

**Problem 333.** A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with  $\alpha = 4$  and  $\beta = 2$ . Find the probability that the wholesaler will sell at least 90% of the stock in a given week.

**Problem 334.** A random variable  $X$  has a *Pareto* distribution if its probability density is given by

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \text{ for } x > 1,$$

where  $\alpha > 0$ .

(a) Show the expected value is

$$\mu = \frac{\alpha}{\alpha - 1}$$

for  $\alpha > 1$ .

(b) Find the  $k$ th moment  $\mu_k$  and show it exists only if  $\alpha > k$

**Problem 335.** A random variable  $X$  has a *Rayleigh* distribution if its probability density is given by

$$f(x) = 2\alpha x e^{-\alpha x^2} \text{ for } x > 0,$$

where  $\alpha > 0$ . Show that for this distribution

(a) The expected value is

$$\mu = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

(b) The variance is

$$\sigma^2 = \frac{1}{\alpha} \left( 1 - \frac{\pi}{4} \right)$$

**Problem 336.** A random variable  $X$  has a *Weibull* distribution if its probability density is given by

$$f(x) = kx^{\beta-1} e^{-\alpha x^\beta} \text{ for } x > 0,$$

where  $\alpha > 0$  and  $\beta > 0$ .

(a) Express the constant  $k$  in terms of  $\alpha$  and  $\beta$ .

(b) Show that the expected value of  $X$  is

$$\mu = \alpha^{-\frac{1}{\beta}} \Gamma \left( 1 + \frac{1}{\beta} \right)$$

Comment: If  $\alpha = 1/\theta$ , for some  $\theta > 0$  and  $\beta = 1$  the Weibull distribution becomes the exponential distribution. If  $\beta = 2$  then the Rayleigh distribution (Problem 335) is obtained. The change of variables  $u = \alpha x^\beta$  will give a gamma density.

**Problem 337.** Resistors used in the construction of an aircraft guidance system have life lengths that follow a Weibull distribution with  $\alpha = 2$  and  $\beta = 10$  (with measurements in thousands of hours).

- (a) Find the probability that the life length of a randomly selected resistor of this type exceeds 5000 hours.
- (b) If three resistors of this type are operating independently, find the probability that exactly one of the three will burn out prior to 5000 hours of use.

**Problem 338.** A function sometimes associated with continuous nonnegative random variables is the *failure rate* (or *hazard rate*) function, defined by

$$r(t) = \frac{f(t)}{1 - F(t)}$$

for a density function  $f(t)$  with corresponding distribution function  $F(t)$ . If we consider the random variable  $X$  as the length of life of a component, then  $r(t)$  is proportional to the probability of failure in a small interval after  $t$ , given that the component has survived up to time  $t$ .

- (a) Show that for an exponential density function,  $r(t)$  is constant.
- (b) Show that for Weibull density function with  $\alpha > 1$ ,  $r(t)$  is an increasing function of  $t$

**Problem 339.** Suppose that  $X$  is a continuous random variable with distribution function given by  $F(x)$  and probability density function  $f(x)$ .

- (a) Show that for a constant  $c$  the conditional probability

$$P(X \leq x | X \geq c) = \frac{F(x) - F(c)}{1 - F(c)}$$

- (b) Show that the function in part **a**) has all the properties of a distribution function.
- (c) If the length of life  $X$  for a battery has a Weibull distribution with  $\alpha = 2$  and  $\beta = 3$  (with measurements in years), find the probability that the battery will last less than four years, given that it is now two years old.

**Problem 340.** A random variable  $X$  has a *Laplace* distribution if its probability density function is given by

$$f(x) = \frac{1}{2\sigma} e^{-\frac{1}{\sigma}|x-\mu|} \text{ for } -\infty < x < \infty,$$

where  $-\infty < \mu < \infty$  and  $\sigma > 0$  are parameters. The probability density function of the Laplace distribution is reminiscent of the normal distribution; however, whereas the normal distribution is

expressed in terms of the squared difference from the mean  $\mu$ , the Laplace density is expressed in terms of the absolute difference from the mean. Consequently, the Laplace distribution has thicker tails than the normal distribution. This distribution is often referred to as “Laplace’s first law of errors”. He published it in 1774 when he noted that the frequency of an error could be expressed as an exponential function of its magnitude once its sign was disregarded.

- (a) Show that the expected value of  $X$  is  $\mu$ .
- (b) Show that the  $n$ th moment of  $X$  about  $\mu$  is zero if  $n$  is odd and is  $n!\sigma^n$  if  $n$  is even.

**Problem 341.** A probability density function for a random variable  $X$  of the form

$$f(x) = \frac{\nu\phi(\nu)}{2\sigma\Gamma(\frac{1}{\nu})} e^{-(\frac{\phi(\nu)}{\sigma}|x-\mu|)^\nu} \text{ for } -\infty < x < \infty,$$

provides a generalization of the normal distribution. Here

$$\phi(\nu) = \sqrt{\frac{\Gamma(3/\nu)}{\Gamma(1/\nu)}}.$$

- (a) Show that if  $\nu = 2$ , then this distribution reduces to the normal distribution.
- (b) Show that if  $\nu = 1$ , then this distribution reduces to the Laplace distribution of Problem 340.

Comment: This generalized normal distribution is very useful in signal processing and in particular modeling of the discrete cosine transform (DCT) coefficients of a digital image.

**Problem 342.** A random variable  $X$  has a *log-normal* distribution if

$$Y = \ln X$$

has a normal distribution. ( $X$  must be nonnegative.) The shape of the log-normal probability density function is similar to that of the gamma distribution, with a long tail to the right. The probability density is given by

$$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \text{ for } x > 0,$$

Since  $\ln x$  is an increasing function,

$$P(X \leq x) = P[\ln X \leq \ln x] = P[Y \leq \ln x],$$

where  $Y$  has a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . The probabilities for random variables with a log-normal distribution can be found by transforming them into probabilities that can be computed using the normal distribution.

- (a) Show that the expected value of  $X$  is

$$\mu = e^{\mu + \frac{\sigma^2}{2}}$$

- (b) Show that the variance of  $x$  is

$$\sigma^2 = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

Comment: The study lognormal distribution was initiated by Galton and McAlister in 1879. They came across this distribution while studying the use of the geometric mean as an estimate of location. Later, Kapteyn (1903) discussed the genesis of this distribution. Often the size distribution of organisms, the distribution of species, the distribution of the number of persons in a census occupation class, the distribution of stars in the universe, and the distribution of the size of incomes are modeled by lognormal distributions. The lognormal distribution is used in biology, astronomy, economics, pharmacology and engineering. This distribution is sometimes known as the *Galton-McAlister* distribution. In economics, the lognormal distribution is called the *Cobb-Douglas* distribution.

**Problem 343.** If the random variable  $X$  has a lognormal distribution with parameters  $\mu$  and  $\sigma^2$ , what is the mode of  $X$ ?

**Problem 344.** If the random variable  $X$  has a lognormal distribution with parameters  $\mu$  and  $\sigma^2$ , what is the median of  $X$ ?

#### 5.2.14 Exercises: Moment-Generating Functions

★ **Problem 345.** Show that for  $\alpha < \beta$ , the moment-generating function of a random variable  $X$  with a uniform probability distribution on the interval  $(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } \alpha < x < \beta \\ 0 & \text{elsewhere} \end{cases}$$

is

$$m(t) = \frac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$$

**Problem 346.** Show that the moment-generating function for a random variable  $X$  with an exponential distribution with parameter  $\theta > 0$

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

is

$$m(t) = \frac{1}{1 - \theta t}$$

**Problem 347.** What is the 479th moment of  $X$  about the origin, if the moment-generating function of  $X$  is  $1/(1 + t)$ ?

**Problem 348.** Show that the moment-generating function for a random variable  $X$  that has a gamma distribution with parameters  $\alpha > 0$  and  $\beta > 0$

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

is the gamma function.

is

$$m(t) = \frac{1}{(1 - \beta t)^\alpha}$$

★ **Problem 349.** Show that,

- (a) if  $X$  is a random variable with moment-generating function  $m(t)$  and  $U = aX + b$ , the moment-generating function of  $U$  is  $e^{bt}m(at)$ .
- (b) if  $X$  has mean  $\mu$  and variance  $\sigma^2$ , use the moment-generating function of  $U$  to derive the mean and variance of  $U$

**Problem 350.** Show that the moment-generating function for a random variable  $X$  with a normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \text{ for } -\infty < x < \infty$$

where  $\sigma > 0$  and  $-\infty < \mu < \infty$ .

is

$$m(t) = e^{\mu t + \sigma^2 t^2/2}$$

**Problem 351.** Identify the distributions of the random variables with the following moment-generating functions:

- (a)  $m(t) = (1 - 4t)^{-2}$
- (b)  $m(t) = 1/(1 - 3.2t)$
- (c)  $m(t) = e^{-5t+6t^2}$

★ **Problem 352.** If the mean and the variance of a certain distribution are 2 and 8, what are the first three terms in the series expansion of the moment generating function?